The Sum Two of Hermitian Operators $\mathcal{A}_i = \mathrm{T}_i + \mu_i$ for Solving the Equations$\{\mathcal{A}_i\mathcal{X} = \mathcal{U}_i\}, i = 1, 2$

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Abstract—In this work we study a new class of equations $(T_i + \mu_i)\mathcal{X} = \mathcal{U}_i, i = 1,2$ including the sum two of Hermitian operators T_i and $\mu_i, i = 1,2$, concerning the kind of spaces are Hilbert. The existence of joint Hermitian solutions to summing two equations of operators has been found under both necessary and sufficient conditions. The $n \times 1$ block's Moore-Penrose inverse of summing two matrix of operators has been studied. Therefore, we present Hermitian solutions of the two equations of operators $(T_i + \mu_i)\mathcal{X}(\mathcal{Q}_i + \mathfrak{m}_i) = \mathcal{U}_i, i = 1,2$, with finding of it's the necessary and sufficient conditions.

Keywords— Operator equation; Sum two Hermitian operators; Common solution; necessary and sufficient conditions; Moore Penrose inverse

1 Introduction

Recently, there has been a lot of interest in the topic Hermitian and positive solutions equations or equations of operators the reason for this is due to its entry into many scientific applications in important and different fields, and also its contributions to the applications we show examples the solvability conditions [1], statistics [2], vibration theory and control system [2, 4], observer design [5] and (see [6-7]).

Researchers have increased interest in studying matrix and operator equation and the system of it, can you see [8-13].

$$\mathrm{T}\mathcal{X} = \mathcal{U} \text{ and } \mathrm{T}_i\mathcal{X} = \mathcal{U}_i, \ i = 1,2, \quad 1$$

In [14], Zhang studied the Hermitian positive semidefinite solution of

$$\mathrm{T}\mathcal{X}\mathcal{V} = \mathcal{U}, \quad 2$$

The Banach or Hilbert spaces was used for matrices and bounded linear operators (see, [13,15-22]), and finding necessary conditions and sufficient conditions (N-SCs) for an existence of a combined solutions, and generalization of two equations

$$\mathrm{T}_i\mathcal{X}\mathcal{V}_i = \mathcal{U}_i, \ i = 1,2, \quad 3$$

Also, the equation's solvability, as follows

$$\mathrm{T}_1\mathcal{X}\mathcal{V}_1 + \mathrm{T}_2\mathcal{X}\mathcal{V}_2 = \mathcal{U}, \quad 4$$

Vosough and Moslehian [23] gave characterizations of the existence and representations of the solutions to the system and restricted the case of operator systems

$$\mathcal{B}\mathcal{X}\mathcal{A} = \mathcal{B} = \mathcal{A}\mathcal{X}, \quad 5$$

In [24], the problems of solutions were expanded system of equations of operators

$$\mathcal{AXB} = \mathcal{C} = \mathcal{BXA}, \quad 6$$

The aim of this work is finding solutions of system with sum two equations of operators

$$\begin{cases} \mathcal{A}_1\mathcal{X} = \mathcal{U}_1 \\ \mathcal{A}_2\mathcal{X} = \mathcal{U}_2 \end{cases}, \text{ where } \left. \begin{matrix} \mathcal{A}_1 = \mathcal{T}_1 + \mu_1 \\ \mathcal{A}_2 = \mathcal{T}_2 + \mu_2 \end{matrix} \right\}, \quad 7$$

and give (N-SCs) for Hermitian solutions (H-s) and for the existence of sum two equations for arbitrary operators.

In addition, we present and prove lemmas for the (H-s) of Eq.(7) and the $n \times 1$ block Moore-Penrose inverse (M-PI) for $[T_1 + \mu_1\ T_2 + \mu_2\ \cdots\ T_n + \mu_n]^t$ matrix of operator, and give (N-SCs) for (H-s) and the existence of Eq.(1) by extend the Dajic and Koliha theorem for sum two arbitrary equations of operators which has not necessarily closed range.

The paper is ordered as follows. In Section 2, we offer an essential concept includes some definitions and properties about unique M-PI and (H-s). In section 3, the main development for summing two equations of operators and the M-PI of a $n \times 1$ block summing two operator of matrixes has been studied. Finally, we explan the conculusion of a new class of sum two of Hermitian are obtained in section 4.

2 Essential Concepts

In this paper, we explain some of the concepts we will needed later in the work. We symbolize H, R and S complex Hilbert spaces and $L(H, R)$ represents of all operators are linear and $B(H, R)$ represents all the bounded operators are linear from H into$R$.
Also,

$$B(H)^{+} = \{T \in B(H) | \langle T(\xi), \xi\rangle \geq 0, \forall \xi \in H\}.$$

and we refer the operators are adjoint of T by T^{*}, and the range of T by K (T) and null space by$N$ (T). Let Q be subspace and closed of H, q_S symbolizes the projection orthogonal onto Q.

Definition (1.1) [25]
The inner inverse is a linear operator of T as T^{-} such that $T \in B(H, R)$ and $T^{-}: D(T^{-}) \subseteq K \to H$ with $K(T) \subseteq D(T^{-})$ and $TT^{-}T = T$.

Definition (1.2) [25]

i) $T \in B(H, R)$ is regular operator if $\exists$an inner inverse $T^{-} \in B(R, H)$.

ii) $\forall T \in B(H, R)$ $\exists$ T^{-} an inner inverse at least one for T but it is not unavoidably bounded.

Definition (1.3) [25]

The generalized inverse T^- of T, if T^- satisfies $T^-TT^- = T^-$, with property T^- is not unique, ∃ a single generalization inverse of T of T and satisfies $(TT^-)^* = TT^-$ and $(T^-T)^* = T^-T$.

Definition (1.4) [26]

The **Moore-Penrose generalized inverse** of T symbolized by T^- is the single generalization inverse of T which satisfying properties, as follows:

1. $TT^-T = T$,
2. $T^-TT^- = T^-$,
3. $(TT^-)^* = TT^-$,
4. $(T^-T)^* = T^-T$.

Now, we offer some properties for $\boldsymbol{T \in B(H, R)}$, as follows [26]

1. An operator $T \in B(H, R)$ has the unique M-PI
$$T^- \in B(R, H) \leftrightarrow T \text{ has closed range.}$$
2. An operator $T \in B(H, R)$ has the unique M-PI
$$T^- \in B(R, H) \leftrightarrow T \text{ is regular.}$$
3. $(T^-)^* = (T^*)^-$.
4. If $T \geq 0$ then $T^- = T^-TT^- = (\sqrt{T}\ T^-)^*(\sqrt{T}\ T^-) \geq 0$.
5. T^-T and TT^- both are projection and $T^-T = P_{\overline{R(T*)}}R(T*)$ and
$$TT^- = P_{\overline{R(T)}}|_{R(T)\oplus R(T)\perp},$$
6. $(TT^*)^- = (T^*)^-T^-$ and $(T^*T)^- = T^-(T^-)^*$,
7. $K(T^-T) = K(T^-) = K(T^*)$ so that $T^-TT^* = T^*$,
8. $K(TT^-) = K(T)$ so that $T^*TT^- = (TT^-T)^* = T^*$.

Theorem 2.1. **[27]**

Let $T, U \in B(H, R)$ and assume T be a closed operator range. Then the equation $T\mathcal{X} = \mathcal{U}$ has a (H-s) $\mathcal{X} \in B(H) \leftrightarrow TT^-U = U$ *and* UT^* *is* Hermitian. Then (H-s) of Eq. (1) is

$$X = T^-U + (I - T^-T)(T^-U)^* + (I - T^-T)Y(I - T^-T)^*,$$

where $Y \in B(H)$ is Hermitian.

$T_1X = U_1$,

$XT_2 = U_2$,

must be a (H-s) $\mathcal{X} \in B(H) \leftrightarrow T_1T_1^-U_1 = U_1$, $U_2T_2^-T_2 = U_2$, $T_1U_2 = U_1T_2$ *and* $T_1U_1^*$, $T_2^*U_2$ *are Hermitian.*

Theorem 2.2 **[11]** *Let* $T \in B(H, R)$, $V \in B(S, H)$ *and* $U \in B(S, R)$. *Then next situations are comparable*

i) $TXV = U$ *is solvable.*

ii) $K(U) \subseteq K(T)$ *and* $K((T^-U)^*) \subseteq K(V^*)$.

iii) $K(U) \subseteq K\ R(T)$ *and* ∃ $\tilde{Y} \in B(H)$ ∋ $\tilde{Y}V = T^-U$.

then ∀ *solution of* $XV = T^-U$ *is also a solution.*

Then ∀ *solution of* $XV = T^{-}U$ *is a solution of* $TXV = U$. *And, for* $\tilde{X} \in B(H)$ $\exists T\tilde{X}V = U$, *we have that* $P_{\overline{R(T^*)}}\tilde{X}$ *is a solution of* $XV = T^{-}U$, *if one of the prior conditions are true.*

3 Main development

In this part, we give and prove the following lemmas about (H-s), also the M-PI of a $n \times 1$ block operator matrix$[T_1\ T_2\ \cdots\ T_n]^t$, and (N-SCs) for the existence of joint (H-s) to the system of sum two operator equations.

Lemma 3.1 *Let* $\mathcal{A}, \mathcal{U}, T$ *and* $\mu \in B(H,R)$ *and assume the equation* $\mathcal{A}\mathcal{X} = \mathcal{U}$, *where* $\mathcal{A} = T + \mu$, *has a solution* $X \in$ B(H) , *then the general form solution of* Eq. (7)is

$$X = (T+\mu)^{-}U + (I - (T+\mu)^{-}(T+\mu))S, \quad \forall S \in B(H). \qquad 8$$

Proof

we assume , Eq. (7) *has a solution, thus by Douglas theorem we have*
$K(U) \subseteq K(\mathcal{A})$, *and* $\mathcal{A} = T + \mu$ *then* $(T+\mu)^{-}U \in B(H)$

$$\text{and } (T+\mu)(T+\mu)^{-}U = U.$$

So $X_0 = (T+\mu)^{-}U$ *is a particular solution of equation* $(T+\mu)X = U$,
and we have
$X = (T+\mu)^{-}U + (I - (T+\mu)^{-}(T+\mu))S, \quad \forall S \in B(H)$,
is the the general for solution of Eq.(7).

Lemma 3.2 Let $\mathcal{A}, \mathcal{U}, T$ *and* $\mu \in B(H,R)$ and assume the equation $\mathcal{A}\mathcal{X} = \mathcal{U}$, where $\mathcal{A} = T + \mu$, *has a* (H-s) $X \in$ B(H) ↔ $K(U) \subseteq K(\mathcal{A})$, *and* $U\mathcal{A}^*$ *is Hermitian. Then* (H-s) to Eq. (7) is

$$X = (T+\mu)^{-}U + (I - (T+\mu)^{-}(T+\mu))\,((T+\mu)^{-}U)^* + \qquad 9$$
$$+(I - (T+\mu)^{-}(T+\mu))S(I - (T+\mu)^{-}(T+\mu)),$$

where $S \in$ B(H) *is Hermitian.*

Proof

The first side

Let $X \in$ B(H) *be a* (H-s) *of Eq.* (7) *then* $K(U) \subseteq K(\mathcal{A})$
and $K(U) \subseteq K(T+\mu)$.
since $\mathcal{A}U^* = (T+\mu)((T+\mu)X)^* = (T+\mu)X^*(T+\mu)^* = (T+\mu)X(T+\mu)^*$,
thus, $(T+\mu)U^*$ *and similary* $U(T+\mu)^*$ *is Hermitian.*
now , we find the general form of (H-s) *of Eq.* (7), *assume Eq.* (7) *has a* (H-s), *then*

$$X_0 = (T+\mu)^{-}U + (I - (T+\mu)^{-}(T+\mu))((T+\mu)^{-}U)^*$$

is a particular (H-s).

if $X \in$ B(H) *be an arbitrary* (H-s) *of Eq.* (7), *then* X- X_0 *is a* (H-s) *of equation* $T\varphi = 0$.
Since φ *has* $(I - \mathcal{A}^{-}\mathcal{A})S(I - \mathcal{A}^{-}\mathcal{A})$,
then $(I - (T+\mu)^{-}(T+\mu))S(I - (T+\mu)^{-}(T+\mu)$ *where* $S \in$ B(H) *and Hermitian, so* X *has the* form of Eq. (9).
in the other word, if $X = (T+\mu)^{-}U + (I - (T+\mu)^{-}(T+\mu))((T+\mu)^{-}U)^{*}$
$+ (I - (T+\mu)^{-}(T+\mu))S(I - (T+\mu) - (T+\mu))$, *where* $S \in$ B(H) *be Hermitian, then it is understandable that* X *is a* (H-s) *of the equation*
$(T+\mu)\mathcal{X} = \mathcal{U}$.

The second side

Let $K(U) \subseteq K(\mathcal{A})$, *and* $U\mathcal{A}^{*}$ *is Hermitian then*
since $\mathcal{A} = T + \mu$, *then by Douglas theorem* $(T+\mu)^{-}U \in B(H)$ *and the equation* $(T+\mu)X = U$ *has a solution.*
since $U\mathcal{A}^{*}$ *is Hermitian, then* $\mathcal{A} = T + \mu$ *and by lemma 1.3*
$X_0 = (T+\mu)^{-}U + (I(T+\mu)^{-}(T+\mu))((T+\mu)^{-}U)^{*}$ *is a particular* (H-s) *of Eq.* (7).■

Lemma 3.3 *Suppose* H, $R_i, i=1,2$ *be Hilbert spaces and* $\mathcal{A}_i \in B(H, R_i) \ni$
$K(\mathcal{A}_i^{*}) \cap K(\mathcal{A}_j^{*}) = \{0\}$
for all $1 \le i \ne j \le n$. *Then* $\begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-} = [(T_1+\mu_1)^{-}\ (T_2+\mu_2)^{-}\ \cdots\ (T_n+\mu_n)^{-}]$.

Proof

Since $K(\mathcal{A}_i^{*}) \cap K(\mathcal{A}_j^{*}) = \{0\}$, *then*
$K([T+\mu]_i^{*}) \cap K([T+\mu]_j^{*}) = \{0\}$, *and*
$\mathcal{N}([T+\mu]_i) = K([T+\mu]_i^{*})^{\perp} \supseteq K([T+\mu]_j^{*}) = K([T+\mu]_j)^{-}$
So , we have $([T+\mu]_i)([T+\mu]_j)^{-} = 0$, $\forall\, i \ne j$, *and* $1 \le i,j \le n$. *and*

$$(T+\mu)(T+\mu)^{-}$$
$$= \begin{bmatrix} (T_1+\mu_1)(T_1+\mu_1)^{-} & 0 & \cdots & 0 \\ 0 & (T_2+\mu_2)(T_2+\mu_2)^{-} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & (T_n+\mu_n)(T_n+\mu_n)^{-} \end{bmatrix},$$

Now, we show that $[(T_1+\mu_1)^{-}\ (T_2+\mu_2)^{-}\ \cdots\ (T_n+\mu_n)^{-}]$ satisfies the Moore pen-rose conditions.

1. $$\begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}$$

$$= \begin{bmatrix} (T_1+\mu_1)(T_1+\mu_1)^{-} & 0 & \cdots & 0 \\ 0 & (T_2+\mu_2)(T_2+\mu_2)^{-} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & (T_n+\mu_n)(T_n+\mu_n)^{-} \end{bmatrix} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}$$

$$= \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix},$$

2. $$\begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-}$$

$$= [(T_1+\mu_1)^{-} \ (T_2+\mu_2)^{-} \ \cdots \ (T_n+\mu_n)^{-}]$$
$$\times \begin{bmatrix} (T_1+\mu_1)(T_1+\mu_1)^{-} & 0 & \cdots & 0 \\ 0 & (T_2+\mu_2)(T_2+\mu_2)^{-} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & (T_n+\mu_n)(T_n+\mu_n)^{-} \end{bmatrix}$$
$$= [(T_1+\mu_1)^{-} \ (T_2+\mu_2)^{-} \ \cdots \ (T_n+\mu_n)^{-}]$$
$$= \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-},$$

3. $$\left(\begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-} \right)^{*}$$

$$= \begin{bmatrix} \left((T_1+\mu_1)(T_1+\mu_1)^{-}\right) & 0 & \cdots & 0 \\ 0 & \left((T_2+\mu_2)(T_2+\mu_2)^{-}\right) & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \left((T_n+\mu_n)(T_n+\mu_n)^{-}\right) \end{bmatrix}^{*}$$
$$= \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-},$$

4. $$\left(\begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix}^{-} \begin{bmatrix} T_1+\mu_1 \\ T_2+\mu_2 \\ \vdots \\ T_n+\mu_n \end{bmatrix} \right)^{*}$$
$$= [(T_1+\mu_1)^{-}(T_1+\mu_1) + (T_2+\mu_2)^{-}(T_2+\mu_2) + \cdots + (T_n+\mu_n)^{-}(T_n+\mu_n)]^{*},$$

$$= \begin{bmatrix} T_1 + \mu_1 \\ T_2 + \mu_2 \\ \vdots \\ T_n + \mu_n \end{bmatrix}^{-} \begin{bmatrix} T_1 + \mu_1 \\ T_2 + \mu_2 \\ \vdots \\ T_n + \mu_n \end{bmatrix}. \blacksquare$$

Example 3.1

Let $\mathcal{Y} = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{bmatrix} = \begin{bmatrix} T_1 + \mu_1 \\ T_2 + \mu_2 \end{bmatrix}$ be a block matrix

where $T_1 + \mu_1 = \left[\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}\right]$,

and $T_2 + \mu_2 = \left[\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\right]$,

Clearly $T_1 + \mu_1$,$T_2 + \mu_2$ and $\mathcal{Y}$ are not invertible, but there are uniquely M-PI for them,

$$\mathcal{Y}^{-} = [(\mathrm{T}_1 + \mu_1)^{-} \quad (\mathrm{T}_2 + \mu_2)^{-}]$$

$$= \left[\left[\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}\right]^{-} \quad \left[\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\right]^{-}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

$K\ ((T_1 + \mu_1)^{*}) \cap K\ ((T_2 + \mu_2)^{*}) = \{0\}$.

Example 3.2

Let $\mathcal{Y} = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \end{bmatrix} = \begin{bmatrix} T_1 + \mu_1 \\ T_2 + \mu_2 \\ T_3 + \mu_3 \end{bmatrix}$ be a block matrix

where $T_1 + \mu_1 = \left[\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$

and $T_2 + \mu_2 = \left[\begin{bmatrix} 0 & \frac{3}{5} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{7}{5} & 0 \end{bmatrix}\right]$

and $T_3 + \mu_3 = \left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$

Clearly $T_1 + \mu_1, T_2 + \mu_2, T_3 + \mu_3$ and $\mathcal{Y}$ are not invertible, but there are uniquely M-PI for them,

$$\mathcal{Y}^{-} = [(\mathrm{T}_1 + \mu_1)^{-} \quad (\mathrm{T}_2 + \mu_2)^{-} \quad (\mathrm{T}_3 + \mu_3)^{-}]$$

$$= \left[\left[\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]^{-} \quad \left[\begin{bmatrix} 0 & \frac{3}{5} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{7}{5} & 0 \end{bmatrix}\right]^{-} \quad \left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]^{-}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

$\mathrm{K}\ ((\mathrm{T}_i + \mu_i)^{*}) \cap \mathrm{K}\ ((\mathrm{T}_j + \mu_j)^{*}) = \{0\}. \quad i, j = 1,2,3.$

4 Conclusion

In this paper, we deduced some developed lemmas for a study a new class of equations$(T_i + \mu_i)\mathcal{X} = \mathcal{U}_i, i = 1,2,$ of sum two of Hermitian operators and the existence of common (H-s) to summing two equations of operators are obtained. Also, M-PI of a $n \times 1$ block summing two matrix of operators has been proved, with the application of illustrative examples that showed the efficiency of the solution for the proposed system.

Reference

[1] Y. H. Liu, Y. G. Tian, and Y. Takane, "Ranks of Hermitian and skewHermitian solutions to the matrix equation AXA∗=B," *Linear Algebra and Its Applications*, vol. 431, no. 12, pp. 2359–2372, 2009.

[2] Y. H. Liu and Y. G. Tian, "More on extremal ranks of the matrix expressions A-BX±X∗B∗ with statistical applications," *Numerical LinearAlgebra with Applications*, vol. 15, no. 4, pp. 307–325, 2008.

[3] D. Hua and P. Lancaster, "Linear matrix equations from an inverse problem of vibration theory," *Linear Algebra and Its Applications*, vol. 246, pp. 31–47, 1996.

[4] Y. H. Liu and Y. G. Tian, "Max-min problems on the ranks and inertias of the matrix expressions A-BXC±(BXC)∗ with applications," *Journal of Optimization Theory and Applications*, vol. 148, no. 3, pp. 593–622, 2011.

[5] L. Dai, *Singular control systems*, Verlag, New York, 1989.

[6] G.R. Duan, *The solution to the matrix equation AV + BW = EV J + R*, Appl. Math. Lett. 17 (2004) 459–474.

[7] P.M. Frank, *Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy- a survey and some new results*, Automatica. 26 (1990) 459–474.

[8] C.G. Khatri, S.K. Mitra, *Hermitian and nonnegative definite solution of linear matrix equation*, SIAM J.Appl. Math. 31 (1976) 579–585.

[9] J. Li, *Positive semidefinite partitioned matrices and a linear matrix equation and its inverse problem*, J.Math. Res. Exposition. 14 (1994) 25–34.

[10] Q. Xu, *Common Hermitian and positive solution to the adjointable operator equations AX = C and XB = D*,Linear Algebra Appl. 429 (2008) 1–11.

[11] M.L. Arias, M.C. Gonzalez, *Positive solutions to operator equations* $AXB = C$, Linear Algebra Appl. 4 33(2010) 1194–1202.

[12] J. K. Baksalary, R. Kala, *The matrix equation* $AXB + CY D = E$, Linear Algebra Appl. 30 (1980) 141–147.

[13] A. Dajic, *Common Solutions of linear equations in a ring, with applications*, Electronic Journal of Linear Algebra. 30 (2015) 66–79.

[14] X. Zhang, *Hermitian nonnegative-definite and positive-definite solution of the matrix equation* $AXB = C$, Appl. Math. E-Notes. 4 (2004) 40–47.

[15] S.K. Mitra, *Common solutions to a pair of linear matrix equation* $A1XB1 = C1$ *and* $A2XB2 = C2$, Proc.Cambridge Philos. Soc. 74 (1973) 213–216.

[16] S.K. Mitra, *A pair of simultaneous linear matrix equations* $A1XB1 = C1$, $A2XB2 = C2$ *and a matrix programming problem*, Linear Algebra Appl. 131 (1990) 107–123.

[17] A. Navarra, P.L. Odell and D.M. Young, *A representation of the general common solution to the matrix equations* $A1XB1 = C1$ *and* $A2XB2 = C2$ *with applications*, Comput. Math. Appl. 41 (2001) 929–935.

[18] A.B. Ozg¨ular, ¨ *The equation* $AXB + CY D = E$ *over a principle ideal domain*, SIAM J.Matrix Anal. Appl. 12 (3) (1991) 581–591.

[19] Q.W. Wang, Z.H. He, *Some matrix equations with applications*, Linear Multilinear Algebra. 60 (2012) 1327-1353.

[20] Q.W. Wang, J.W. Woude and H.X. Change, *A system of real quaternion matrix equations with applications*,Linear Algebra Appl. 431 (2009) 291–2303.

[21] Q.W. Wang, *A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity*, Linear Algebra Appl. 384 (2004) 43–54.

[22] J.W. Woude, *On the existence of a common solution X to the matrix equations* $AiXBj = Cij$, $(i; j)$ $2\ \Gamma$, Linear Algebra Appl. 375 (2003) 135–145.

[23] Vosough M, Moslehian M S. Solutions of the system of operator equations AXB = B = BXA via ∗-order Electronic J Linear Algabra, 2017, 32: 172–183.

[24] Arias M L, Gonzalez M C. Positive solutions to operator equations AXB = C. Linear Algebra Appl, 2010,433(6): 1194–1202.

[25] A. Ben-Israel, T.N.R. Greville, *Generalized Inverses, Theory and Applications*, Springer-Verlag, New York, vol 15, 2003.

[26] M.Z. Nashed, *Inner, outer, and generalized inverses in Banach and Hilbert space*, Numer. Funct. Anal. Optim. 9 (1987) 261–325.

[27] A. Dajic, J.J. Koliha, *Positive solutions to the equations AX = C and XB = D for Hilbert space operators*, J.Math. Anal. Appl. 333 (2007) 567–576.

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