Computations Stiefel-Whitney classes of real representations for non-prime power groups

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Abstract— In the current study, the researchers conduct computation of the Stiefel- Whitney classes of real representations of non-prime power groups such as Mathieu groups, symmetric groups, alternating groups and Janko groups. Stiefel- Whitney classes are conducted by using orbit polytope convex hull and steenrod squares of the non-prime power groups whick we get it by using the sylow theorm and with vector in n-dimention our computation using the HAP package (Homological Algebra Programming) of the GAP(general Algebra Programming) Programming software in other word this paper demonstrates how the underlying techniques can be used to get the SW-classes of the real representation for non-prime power groups **.**

Keywords— Cohomology operation, Steenrod square, Cup-i product, SW-classes .

1 Introduction

In this paper, Steenrod squares with cupi- product, orbit polytopes and the $Thom\ isomorphism$ are used to compute the Stiefel − Whitney classes (pithiness SW-classes) comparable to a real representation $\rho : G \rightarrow O_n(\mathbb{R})$ of a non-prime power groups. The main obstacle was how to calculate the steenrod squares on cohomology rings of non-prime power groups, which were computed by Mahmood [1]. However, if the cohomology ring of the sylow subgroup of any non-prime power groups is homeomorphic to finite groups then they are less than or equal to 128 groups. The SW-classes comparable to a real representation $\rho : G \rightarrow O_n(\mathbb{R})$ of a finite groups studies were taken up Ellis [2,3], Guillot [4] and Daher [5]

1.1 Definition [**2, 5**] A segment connecting any two points $a, b \in X$, a subset $X\ of\ \mathbb{R}^n$ is called Convex, and it implies that for every pair and $\lambda \in [0,1]$, which is real number, one has $\lambda a + (1 - \lambda)b \in X$. The finite set $X = \{a_1, a_2, \dots, a_d\}$ of points in $\mathbb{R}^n$ called *convex hull* and denoted *conv*(*X*),

$$conv(X) = \{\sum_{i=1}^{d} \lambda_i a_i : a_i \epsilon X, \lambda_i \geq 0, \sum_{i=1}^{d} \lambda_i = 1\}$$

1.2 Definition1 [6] CW-complex is an abbreviation of what means "closure-finite weak topology" which is obtained by taking the union of a sequence of topological spaces

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

Each space X_n is formed from X_{n-1} such that $X_n = X_{n-1} \coprod_{\propto} e^n_{\propto}$ where each $e^n_{\propto}$ is an open n-disk. X is called CW-complex if $X = \bigcup_n X_n$

1.3 Definition [5] if L and K are two R-modules over any ring R Their tensor product is an R-modules such that $L \otimes_R K = \{l \otimes k$, for some $l \in L$ and $k \in K\}$ subject to the following relations

- $(l_1 + l_2) \otimes k = (l_1 \otimes k) + (l_2 \otimes k)$
- $l \otimes (k_1 + k_2) = (l \otimes k_1) + (l \otimes k_2)$
- $r.l \otimes k = l \otimes r.k$, $\forall r \in R, l_1, l_2 \in L$ and $k_1, k_2 \in K.$

1.4 Definition [7] To construct a contractible CW-space EG, there are various ways for any group G, where G acts freely; the action changes the cell when referring to EG as a total space for G. The base space $BG = EG/G$ is a quotient space. A classification space for G is produced by omitting the action. We have a free $\mathbb{Z}$G − resolution which is the cellular chain complex $C_*(EG)$ and $C_*(BG) = C_*(EG) \otimes_{\mathbb{Z}G} \mathbb{Z}$. Thus, $H^*(BG, \mathbb{Z}_p) = H^*(G, \mathbb{Z}_p)$.

1.5 Definition [8] let $X \xrightarrow{\Delta} X \times X$ be the diagonal map $C_{p+q}(X) \xrightarrow{d_\Delta} C_{p+q}(X \times X) \xrightarrow{\varphi} C_p(X) \otimes C_q(X) \xrightarrow{u \otimes v} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}$

$u \smile v$(c) = $(u \otimes v)\varphi \circ d_\Delta$(c) = $u \otimes v$ (φ (d_Δ (c))) , c ∈ C_{p+q}(X). As a result, the cup product is the homomorphism $\smile : H^p(X) \otimes H^q(X) \to H^{p+q}(X)$ described as u ⌣ v = d∗(u × v).
(Coefficients belong to any commutative ring with unity). One direct effect of the cross product rules is that $u \smile v = (-1)^{pq} u \smile v$, in which p and q are the degrees of u and v, respectively. The general case of cup product is the cup − i product such that they are equal if i = 0, [$i.e$: = $u \smile_0 v = u \smile_i v$].
For every ($i \geq 0; i \in \mathbb{Z}$) a $\mathbb{Z}$ − linear cup − i product

$$C^p(R^G_*) \otimes_{\mathbb{Z}} C^q(R^G_*) \to C^{p+q-1}(R^G_*), \quad \bar{u} \otimes \bar{v} \mapsto \bar{u} \smile_i \bar{v} \ldots\ldots (1)$$

is explained by the formula $(\bar{u} \smile_i \bar{v})(c) = (\bar{u} \otimes \bar{v})\phi_{p+q}(k^i \otimes c)$ ... (2) $for\ c \in R^G_{p+q-i}$

1.6 Theorem [4,5] The operation $C^n(R^G_*) \to C^{2n-i}(R^G_*)$, $\bar{u} \mapsto \bar{u} \smile_i \bar{u} \ldots$ (3)
Induces a homomorphism $Sq^i : H^n(\mathrm{G}, \mathbb{Z}_2) \to H^{2n-i}(\mathrm{G}, \mathbb{Z}_2)$ … (4)
The homomorphism $Sq^i = sq^{n-i} : H^n(\mathrm{G}, \mathbb{Z}_2) \to H^{n+i}(\mathrm{G}, \mathbb{Z}_2)$ … (5)

2 Construct Stiefel- Whitney classes

In this section, we discuss the method of computation or construction SW-classes. We have some steps to find them.

1) Let $\rho: G \to O_n(\mathbb{R})$ or $GL_n(\mathbb{R})$ be an any (orthogonal) matrix representation of a finite group G.
2) Take $v \in \mathbb{R}^n$ be any vector.
3) Let $\Omega(g, v) = \{g.v: g \in G\}$ represents convex hull of $\Omega(g, v)$ in $\mathbb{R}^n$
4) The vertices of the orbit polytope $P = P(\rho, v) = \rho(g)v = g.v = \Omega(g, v)$, convex hull of $\Omega(g, v)$ in $\mathbb{R}^n$,

We have the convex polytope by choosing a vector $v \in \mathbb{R}^n$,

$$P = P(\rho, v) = conv\{\rho(g).v : g \in G\}$$

is a CW-space homotopy equivalent to a ball B^n and P^{n-1} is equivalence to a sphere S^{n-1}. $C_*(P)$ is cellular chain complex of the polytope P which is a complex of $\mathbb{Z}$G-modules (may be not free). "A contractible $m-$dimensional and G acts on it by permuting cells" is the polytope P, and "the action endows P_*^G with structure of a $\mathbb{Z}G-$resolution of $\mathbb{Z}$."

$$C_*(P): 0 \to C_K(P) \to C_{K-1}(P) \to \cdots \to C_0(P)$$

And $\quad C_*(P^{K-1}): 0 \to C_{K-1}(P) \to C_{K-2}(P) \to \cdots \to C_0(P)$

R_*^G *is* $\mathbb{Z}G-$resolution of $\mathbb{Z}$

Then the tensor product $R_*^G \otimes C_*(P)$ if (non- free) $\mathbb{Z}$G –resolution with

$g(a * b) = ga * gb$, where $P = P(\rho, v) = ball$, $P^{n-1} = sphere$.

And we have the equivalence $\quad R_*^G \otimes C_*(P) \cong EG \times B^n$

$$R_*^G \otimes C_*(P^{n-1}) \cong EG \times S^{n-1}$$

In general the module $C_K(P)$ is not always free for $1 \le m \le K$, such that K represents the dimension of P. The free abelian group on one generator is the module $C_K(P)$ given a possibly non-trivial G action, we denote $C_K(P)$ by $\mathbb{Z}^{\epsilon}$. For the i^{th} orbit of p-cells, let $G_*^k \subseteq G$ represent "the stabilizer group of some cell in the orbit", and let $R_*^{G_i^k}$ designate some "free $\mathbb{Z}G_i^k-$resolution of $\mathbb{Z}$ ".

$$R_*^{G_i^k} : \ldots \to R_2^{G_i^k} \to R_1^{G_i^k} \to R_0^{G_i^k}$$

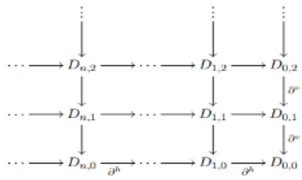
The polytope P has dimension $K = dim(P)$. In addition, $R^{G_i^K} = R_*^G$ denotes a free $\mathbb{Z}G$ –resolution, when the unique K-dimensional cell has stabilizer group $G_1^K = G$. The direct sum of $\mathbb{Z}G$-modules is a module $C_p(P)$.

$$C_p(P) = \oplus_{1\le i\le d_k} \mathbb{Z}G \otimes_{\mathbb{Z}G_i^k} \mathbb{Z}$$

By defining $D_{p,q} \coloneqq \oplus_{1\le i\le d_k} \mathbb{Z}G \otimes_{\mathbb{Z}G_i^k} R_q^{G_i^k}$ to obtain a free $\mathrm{Z}G$-resolution

$$D_{p,*} : \ldots \to D_{p,q} \to \cdots \to D_{p,1} \to D_{p,0}$$

of the module $C_p(P)$. The boundary maps in $C_*(P)$ induce chain maps $\partial h : D_{p,q} \to D_{p-1,q}$ to yield a diagram $D_{*,*}$ of free $\mathbb{Z}G$-module



The vertical maps $\partial_v : D_{p,q} \to D_{p,q-1}$ in $D_{*,*}$ Satisfy $\partial^v \partial^v = 0$.

$$\ldots \to R_2^G \otimes \mathbb{Z} \xrightarrow{\alpha} R_1^G \otimes \mathbb{Z} \xrightarrow{\alpha} R_0^G \otimes \mathbb{Z}$$

$$\ldots \to C_2(P) \xrightarrow{\partial} C_1(P) \xrightarrow{\partial} C_0(P)$$

$$\begin{array}{ccccc}
\ldots \to C_2(P) \otimes R_2^G \otimes \mathbb{Z} & \xrightarrow{\partial_1 \otimes I} & C_1(P) \otimes R_2^G \otimes \mathbb{Z} & \xrightarrow{\partial_0 \otimes I} & C_0(P) \otimes R_2^G \otimes \mathbb{Z} \\
I \otimes \alpha_1 \downarrow & & I \otimes \alpha_1 \downarrow & & I \otimes \alpha_1 \downarrow \\
\ldots \to C_2(P) \otimes R_1^G \otimes \mathbb{Z} & \xrightarrow{\partial_1 \otimes I} & C_1(P) \otimes R_1^G \otimes \mathbb{Z} & \xrightarrow{\partial_0 \otimes I} & C_0(P) \otimes R_1^G \otimes \mathbb{Z} \\
I \otimes \alpha_0 \downarrow & & I \otimes \alpha_0 \downarrow & & I \otimes \alpha_0 \downarrow \\
\ldots \to C_2(P) \otimes R_0^G \otimes \mathbb{Z} & \xrightarrow{\partial_1 \otimes I} & C_1(P) \otimes R_0^G \otimes \mathbb{Z} & \xrightarrow{\partial_0 \otimes I} & C_0(P) \otimes R_0^G \otimes \mathbb{Z}
\end{array}$$

$$\xrightarrow{\partial_0 \otimes I \oplus I \otimes \alpha_1 \oplus \partial_1 \otimes I \oplus I \otimes \alpha_0} \quad C_0(P) \otimes R_1^G \otimes \mathbb{Z} \quad \xrightarrow{\partial_0 \otimes I \oplus I \otimes \alpha_0}$$

Now, Hom($R_*^G \otimes C_*(P)$)=E_*^G, and it is denoted by E_*^G, that can construct a second $\mathbb{Z}G$-resolution. This second resolution has

$$E_n^G = \bigoplus_{\substack{p+q=n \\ 0\le p\le D}} D_{p,q} = \bigoplus_{\substack{p+q=n \\ 0\le p\le D}} \bigoplus_{1\le i\le d_p} R_q^{G_i^k} \oplus_{ZG_i^k} \mathbb{Z}G$$

We can define $F_*^G < E_*^G$ as a "ZG − subchain complex "by setting

$$F_n^G = \bigoplus_{\substack{p+q=n \\ 0\le p\le D-1}} D_{p,q} = \bigoplus_{\substack{p+q=n \\ 0\le p\le D-1}} \bigoplus_{1\le i\le d_p} R_q^{G_i^k} \oplus_{ZG_i^k} \mathbb{Z}G$$

We let $\bar{R}_* = E_*^G/F_*^G$ Then

$$\overline{R_n} = \begin{cases} 0 & n<K \\ R_{n-K}^G \otimes_{\mathbb{Z}} \mathbb{Z}^{\epsilon} & n \ge K \end{cases} \qquad \dots(3)$$

An Addition, there is a short exact sequene .

$$F_*^G \rightarrowtail E_*^G \twoheadrightarrow \overline{R_*} \qquad \dots(4)$$

Taking $\mathbb{Z}_2$ as a field and applying the contravariant functor $\mathrm{Hom}_{\mathbb{Z}_2\mathrm{G}}(\text{-} , \mathbb{Z}_2)$ we acquire "the short exact sequence of cochain complexes"

$$Hom_{\mathbb{Z}_2G}(\overline{R_*}, \mathbb{Z}_2) \rightarrowtail Hom_{\mathbb{Z}_2G}(E_*^G, \mathbb{Z}_2) \twoheadrightarrow Hom_{\mathbb{Z}_2G}(F_*^G, \mathbb{Z}_2) \qquad \dots(5)$$

The explanation behind working over $\mathbb{Z}_2$ is that we can ignore the action of G on $\mathbb{Z}^{\epsilon}$. In equation (5) the inclusion of cochain complexes induces cohomology homomorphisms

$$\rho^t : H^t(G, \mathbb{Z}_2) \to H^{t+K}(G, \mathbb{Z}_2) \qquad \dots(6)$$

The cohomology homomorphism ρ^k is only dependent on "ρ" and to a certain extent on the vector u. The cochain complexes in (5) as

$$L^* = Hom_{\mathbb{Z}_2G}(\bar{R}_*, \mathbb{Z}_2)\ , M^* = Hom_{\mathbb{Z}_2G}(E_*^G, \mathbb{Z}_2)\ , M^*/L^* = Hom_{\mathbb{Z}_2G}(F_*^G, \mathbb{Z}_2)$$

The relative cohomology can be define by setting

$$H^n(M^*, L^*) = H^n\left(\frac{M^*}{L^*}\right) \qquad \dots(8)$$

in low dimension $n < K$

$$H^n(G,\mathbb{Z}_2) = H^n(M^*, L^*) \qquad \dots (9)$$

We obtain the isomorphism from (3)

$$\tau\colon\ H^n(\mathrm{G},\mathbb{Z}_2) \cong \mathrm{H}^{\mathrm{n+K}}(M^*, L^*)\ , n \geq 0 \qquad \dots (10)$$

And $H^n(M^*, L^*) = 0$, where $n < K$. The isomorphism τ is called the Thom isomorphism (for more information see [9, 6]) .

The cohomology cup product is defined as

$$\smile\colon H^p(M^*) \times H^q(M^*) \to H^{p+q}(M^*) \quad \dots (11)$$

A cup product can extend to

$$H^p(K^*, L^*) \otimes_{\mathbb{Z}_2} H^q(M^*, L^*) \to H^{p+q}(M^*, L^*) \qquad \dots (12)$$

Taking ψ to signify the non-zero component of $H^K(M^*, L^*)$, it is possible to demonstrate that the formula (2) yields the Thom isomorphism in terms of (12).

$$\tau(x) = \psi \smile x \quad \dots (13)$$

And by using the properties of a cup-i product

$$\smile_i\colon M^p \otimes_{\mathbb{Z}_2} M^q \to M^{p+q-i}.$$

We can restrict $\mathrm{L}^* < \mathrm{M}^*$ be the subcochain complex producing a cup-i product

$$\smile_i\colon L^p \otimes_{\mathbb{Z}_2} L^q \to L^{p+q-i} \qquad \dots (14).$$

We can use properties of Steenrod square and formula (5) in order to define Steenrod squares on relative cohomology, with the restricted cup-i product.

$Sq^i\colon H^n(M^*, L^*) \to H^{n+i}(M^*, L^*)$

2.1 Definition [5] the i^{th} *SW- class* $\omega_i \in H^i(G, \mathbb{Z}_2)$ is described by the formula

$$\omega_i = \tau^{-1}(Sq^i(\psi))$$

If we have the representation $\rho : G \to O_n(R)$ of a finite group G, and $v \in \mathrm{R}^n$ as a vector, or similarly $\psi \smile \omega_i = Sq^i(\psi)$, *where* $0 \neq 1 \in H^0(M^*, L^*) \cong \mathbb{Z}_2$.

In particular $\omega_0 = 1$ *and* $\psi = \tau(1)$.

The total SW- class can be explained as $\omega(\rho, v) = \omega_0 + \omega_1 + \cdots + \omega_K$

We recommend reading [10, 9] for a theoretical explanation of SW-classes. A direct computer implementation of specification (2.1) may not be practical because "the size of the resolution E_*^G underlying the definition". Although, the homomorphism ρ^0 : $H^0(M^*, L^*) \to H^K(G, \mathbb{Z}_2)$ it is feasible to calculate $\rho^0(1) = \psi$. If it occurs that $\rho^0(1)$ is non-zero, then ρ^0 is an isomorphism. The Naturality formula is

$$\begin{array}{ccc} H^0(M^*, L^*) & \xrightarrow{\rho^0} & H^K(G, \mathbb{Z}_2) \\ Sq^i \downarrow & & \downarrow Sq^i \\ H^i(M^*, L^*) & \xrightarrow{\rho^i} & H^{K+i}(G, \mathbb{Z}_2) \end{array}$$

$$\rho^i(Sq^i(1)) = Sq^i(\rho^0(1))$$

Hence, any solution ω_i to $\psi \cup \omega_i = Sqi(\psi)$ yields $\rho^0(1) \cup \omega_i = Sq^i\big(\rho^0(1)\big)$. is totally within the ring $H^*(M) = H^*(G, \mathbb{Z}_2)$. This final formula can be used to calculate the multiple $\rho^0(1) \cup \omega_i$. No doubt, this multiple contains helpful information when $\rho^0(1) \neq 0$.

3 Computing method SW- classes of real representations of non-prime power groups

We previously learned the method of computing method SW-classes. As for the non-prime power groups the method revolves around how to compute Steenrod square for this type of groups. To compute Steenrod square on cohomology rings of non $-$ prime power groups are less than or equal to 128 group. However, if the cohomology ring of the sylow subgroup of the given group known, we could calculate the steenrod square on cohomology rings of several non-prime groups whose sylow subgroup is order≤ 128. For example, the group $G_{8,5}$ of order 8 and number 5 belong to $syl_2(G)$. Structure Description is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where G isomorphic to J_1 (janko group) is a realizable group of type J_1. Also, $G_{16,8}$ the group of order 16 and number 8 in GAP library belong to $syl_2(G)$ where Structure Description is QD_{16} ,where $G \cong M_{11}$. Is means that $G_{16,8}$ is a realizable group of type M_{11}.We applied the above for all groups, and it is realizable .

3.1 Example Consider the group G= $Syl_2(M_{11})$ of order 16 and number 5, emerging as the sylow $2-$ subgroup of the M_{11}. let ρ signify the representation that transfers each permutation $g \in G$ to 11×11 permutation matrix $\rho(g)$ and let $v = (1,2,3,4,5,6,7,8,9,10,11) \in R^{11}$. The total SW-class will be calculated at the next GAP session.

GAP session

```
gap > G := SylowSubgroup(MathieuGroup(11), 2);
Group([ (2,4)(5,9)(6,11)(7,8), (3,10)(5,9)(6,8)(7,11), (2,5)(3,10)(4,9)
(7,8), (2,6,9,8,4,11,5,7)(3,10) ])
gap > A := Mod2SteenrodAlgebra(G, 12);;
gap > gens := ModPRingGenerators(A);
[ v.1, v.2, v.3, v.6, v.8 ]
gap > List(gens, A!.degree);
[ 0, 1, 1, 3, 4 ]
gap > rho := PermToMatrixGroup(G);
[ (2,4)(5,9)(6,11)(7,8), (3,10)(5,9)(6,8)(7,11), (2,5)(3,10)(4,9)(7,8),
  (2,6,9,8,4,11,5,7)(3,10) ] ->
[ [ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
     [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]……
gap> v:=[1,2,3,4,5,6,7,8,9,10,11];;
     gap > sw:
     = FundamentalMultiplesOfStiefelWhitneyClasses(rho, v, A, true);
[ [ 0*v.1, v.1 ], [ 0*v.1, v.3, v.2, v.2+v.3 ], [ 0*v.1, v.5, v.4, v.4+v.5 ], [ 0*v.1,
v.7, v.6, v.6+v.7 ],[ 0*v.1, v.10, v.9, v.9+v.10, v.8, v.8+v.10, v.8+v.9,
v.8+v.9+v.10 ],…..
gap > TotalStiefelWhitneyClass: =
sw[1][2]+sw[2][2]+sw[3][2]+sw[4][2]+sw[5][2]+sw[6][2]+sw[7][2]+sw[8][2]+s
w[9][2];
v.1+v.3+v.5+v.7+v.10+v.14+v.18+v.22+v.27
gap > PrintAlgebraWordAsPolynomial(A, TotalStiefelWhitneyClass);
v.1 + v.3 + v.2*v.2 + v.2*v.3 + v.2*v.2*v.2 + v.2 * v.2 * v.2 * v.2 +
v.2*v.2*v.2*v.2*v.2 + v.6*v.2*v.2 +  v.2*v.2*v.2*v.2*v.2*v.2 + v.6*v.2*v.2*v.2 +
v.2*v.2*v.2*\ v.2*v.2*v.2*v.2 + v.2*v.2*v.2*v.2*v.2*v.2*v.2*v.2
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