An Innovative Approach to Evaluating the Approximate Solutions Concerning the Fractional Sharma-Tasso-Oliver Equation Comparison

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Abstract— In this research, we introduce an improved analytical approximation technique for addressing the time-fractional Sharma-Tasso-Olever problem. To manage nonlinear fractional differential equations that emerge in numerous physical phenomena, we establish an alternative basis for the Laplace Residual Power Series approach (LRPSA). The generalized Taylor series equation and residual functions form the foundation of this strategy. The proposed solution yields positive outcomes. The dependability, efficiency, and simplicity of the suggested method are showcased across all categories of fractional nonlinear problems encountered in technological and scientific domains. Two examples are given to illustrate the effectiveness of the proposed approach in solving various kinds of fractional ordinary differential equations. A comparison with other techniques such as RPS, VIM, HPM reveals that our method produces favourable and efficient results.

Keywords— Calculus of fractions, Sharma-Tasso-Olever fractional equation, Remaining power sequence, Laplace remaining power series, Caputo's fractional differential.

1 Introduction

Fractional calculus (FC) represents a subfield of applied mathematics concerned with derivatives and integrations of real or complex orders, also known as non-Newtonian calculus or extended calculus. The inception of FC can be traced back to 1695 when Leibniz contemplated the possibility of a derivative with an order of $\frac{1}{2}$ in a famous letter [1,2]. FC has garnered importance in various domains, including physics, biochemistry, biology, technology, viscoelasticity, operations research, optical fibers, communication, and finance [3]. While not all methods are ubiquitously utilized, there exists an array of approaches for defining fractional differential equations.

There exist several approaches to defining fractional derivatives, but not all are commonly utilized. The most frequently employed fractional derivatives involve fractional rank with respect to Conformable operators (CO), Atangana-Baleanu (A-B), Riemann-Liouville (R-L), and Caputo fractional derivatives (CFD) [6, 7, 8, 9, 10]. In certain scenarios, fractional derivatives are preferred to integer-order derivatives for modeling purposes, as they can model and assess complex systems with superior non-linear processes and higher-rank dynamic. This is due to two primary factors. Firstly, instead of being restricted to an integer order, we have the freedom to select any order for derivative operators. Non-integer type derivatives, depending on previous and local circumstances, are advantageous whenever the systems have such a long-term memory.

A multitude of numerical and analytical methodologies, such as the homotopy method, Laplace transforms, the Pade-variational iteration method, the Adomian decomposition method, and others [11-21], have been created to tackle fractional differential equations. However, these approaches are encumbered by the necessity of extensive and intricate computations. As such, they must be coupled with a transform operator. The authors of [22] innovatively combined the Laplace transform with the residual power series method (RPS) to introduce real and series solutions to both linear and nonlinear FDEs. This new approach, christened as the Laplace-residual power series method (LRPS), was implemented to formulate series solutions for various FDEs. The LRPS methodology is not dependent on fractional derivation for determining the coefficients of the series, unlike the RPS, but relies instead on the concept of limit, facilitating fewer computations to generate the coefficients, as opposed to the residual power series method. The current methodology is speedy, requires minimal computer memory, and is impervious to computational round-off errors. Moreover, this methodology calculates the coefficients of the power series via a string of equations with multiple variables, indicating that the present approach has swift convergence.

In the pursuit of detecting both approximations and exact results for time fractions from Sharma-Tass-Olver PDEs involving unknown parameters, our study employed a proprietary hybrid method named LRPS. This new technology combines RPS and Laplace conversion technology, while also showcasing the graphical significance of different values of fractional order derivatives. The accuracy, speed, and imperviousness to repeating errors make this method a standout, as it does not consume much memory storage space or processing time.

To explore the realms of the nonlinear fractional Sharma-Tasso-Oliver formula and its integral role in uncovering nonlinear phenomena, our study employed LRPS in the guise of nonlinear time fractions. We conducted a comparative analysis of this approach with other techniques, including Variation Iteration Approach, Adomian Decomposition, Homotopy Perturbation Method, and Residual Power Series Technique of the Sharma-Tasso-Oliver formula. The definition of FSTOF stands as follows:

$$D_{t*}^{\alpha}u + 3\rho u_x^2 + 3\rho u^2 u_x + 3\rho u u_{xx} + \rho u_{xxx} = 0 \,, \eta > 0, 0 < \alpha \leq 1, \tag{1}$$

where D_{t*}^{α} is Caputo derivative of order $0 < \alpha \leq 1$, ρ is constant and u is analytical function and t^* is time variable.

The given expression involves the Caputo derivative of order $0 < \alpha \leq 1$, denoted by D_{t*}^{α} where ρ is a constant and u is an analytic function.

The following section passionately delves into the intricacies of our research. The introduction of Section 2 acquaints us with the essential principles and theories in fractional calculus and Laplace transforms that are indispensable in obtaining our desired outcomes. The algorithm that goes by the name of LRPS has been expounded upon in the third section. In Section 4, numerical examples are presented and the solutions are compared with the results obtained from other established methods. The findings of this comparison are discussed. Section 5 concludes the study by summarizing the results and presenting the conclusions.

2 Fractional extension via laplace spaces

The operators of fractional derivatives exhibit non-local characteristics, as they are defined using integrals. Hence, the time-fractional derivative captures information about the function's earlier stages and therefore, demonstrates a memory effect. Such derivative operators account for the history and non-local distributed effects, which are critical for more precise and accurate representations, as well as an understanding of complex and dynamic system behavior.

In this section, we introduce the fractional calculus in Caputo's sense and the Laplace transform, which are indispensable in creating the LRPS solution for the fractional neutron diffusion equations with one delayed neutron group. We present some fundamental concepts and ideas to aid in this endeavor.

Definition 2.1 [23]. The definition of the Caputo derivative of order α of $\psi(x,\eta)$ sense involves the utilization of a time-fractional component:

$$D_{t*}^{\alpha}\psi(x,t) = J^{m-\alpha}\partial_{\eta}^{m}\psi(x,t),\ m-1<\alpha\le m, x\in I, t>0, \tag{2}$$

whereas D_{t*}^{α} represents the operator for time-fractional Caputo derivative of order α; whereby $m\in$ N; I signifies an interval; and J_{η}^{β} denotes the operator for time-fractional Riemann–Liouville integral of order β, which is explicitly defined as:

$$J_{\eta}^{\beta}\psi(x,t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\sigma)^{\beta-1}\psi(x,\sigma)d\sigma, \beta>0, t>\sigma\ge 0, \text{ and}$$

$$\psi(x,\sigma) = 0, \beta = 0, \tag{3}$$

In the ensuing lemma, we shall divulge specific characteristics of the operator D_{t*}^{α} that are crucial for the continued progress of our research. Peruse [23] for further facets that may pique your interest.

Lemma 2.2. [23]. For $t\ge 0, \eta>-1,, \sigma\in\mathbb{R}$, and $\alpha\in(m-1,m]$.

1. $D_{t*}^{\alpha}\sigma = 0.$
2. $D_{t*}^{\alpha}J_t^{\alpha}\psi(x,t) = \psi(x,t).$
3. $J_t^{\alpha}D_{t*}^{\alpha}\psi(x,t) = \psi(x,t) - \sum_{m=0}^{n-1}\frac{\partial_t^m\omega(x,0^+)}{m!}t^m.$
4. $D_t^{\alpha}t^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)}t^{\eta-\alpha}.$

Definition 2.3. [24] The Laplace transform of a continuous function $\psi(x,t)$, which is defined on a specific interval $I\times[0,\infty)$ and possesses an exponential order of μ a certain magnitude, is expressed by a distinct symbol and determined in a prescribed manner as follows:

$$\Psi(x,s) = \mathcal{L}\{\psi(x,t)\} = \int_0^{\infty} e^{-st}\psi(x,t)dt,\ s>\alpha, \tag{4}$$

whereas the exquisite and intricate inverse Laplace transform of the function $\Psi(x, s)$ is elegantly and precisely defined as follows:

$$\psi(x,t) = \mathcal{L}^{-1}\{\Psi(x,s)\} = \int_{c-i\infty}^{c+i\infty} e^{s\eta}\Psi(x,s)ds,\ c = \mathrm{Re}(s) > c_0, \tag{5}$$

Now, as illustrated by the ensuing outcome, we introduce a fresh fractional expansion which will act as the cornerstone for formulating a LRPS resolution to the partial differential equations (PDEs).

Proposition 2.4. [24] Let $w(x, \eta)$ be a continuous function defined on the interval $I \times [0, \infty)$ and possessing exponential order μ. Assuming that the function is characterized by the subsequent fractional expansion $\Psi(x, s) = \mathcal{L}\{\psi(x, t)\}$, we will proceed with our analysis.

$$\Psi(x,s) = \sum_{m=0}^{\infty} \frac{\lambda_m(x)}{s^{1+m\alpha}},\ 0 < \alpha \leq 1, x \in I, s > \mu, \tag{6}$$

Hence, $\lambda_m(x) = D_t^{m\alpha} w(x, 0)$.

Remark 2.5.[24] Applying the inverse Laplace transform to Eq. (6) yields:

$$\psi(x,t) = \sum_{m=0}^{\infty} \frac{D_t^{\alpha}\psi(x,0)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, t \geq 0, \tag{7}$$

The convergence of the series in the expansion (6) is ensured by the conditions presented in the subsequent theorem.

Theorem 2.4 (El-Ajou, 2021) Let $\psi(x, t)$ be a continuous function defined on the interval $I \times [0, \infty)$ and possessing exponential order μ. Moreover, suppose that $\Psi(x, s) = \mathcal{L}[\{\psi(x, t)\ ]$ it can be expressed as the fractional expansion indicated in Theorem 2.3. In the event that on $\left|s\mathcal{L}\left[\mathfrak{D}_t^{(n+1)\alpha}\psi(x,t)\ \right]\right| \leq M(x)$, on $I \times (\mu, \kappa]$ where $0 < \alpha \leq 1$, it follows that the residual term $R_n(x, s)$ associated with the fractional expansion presented in Eq. (6) is bounded by the subsequent inequality:

$$|R_m(x,s)| \leq \frac{M(x),}{s^{(m+1)\alpha+1}}, x \in I, \delta < s \leq \gamma. \tag{8}$$

3. The LRPS technique is utilized to derive creative solutions for the Time-Fractional Sharma-Tasso-Olever Formula.

we shall employ the fractional Sharma-Tasso-Olever formula, with due consideration to time, to showcase the efficacy of the LRPS methodology in generating a sequence of solutions to the FSTOF as defined in Eq. (1) with initial condition:

$$u(x,0) = \zeta(x), \tag{9}$$

In the initial stages, it is recommended to employ the LT for (1), thereby yielding the desired outcome.

$$\mathcal{L}[D_t^{\alpha}u(x,t) + 3\lambda u_x^2(x,t) + 3\lambda u^2(x,t)u_x(x,t) + 3\lambda u(x,t)u_{xx}(x,t) + \lambda u_{xxx}(x,t)]$$

$$= \mathcal{L}[0],\ t \in\ I \times [0, \infty]. \tag{10}$$

We can derive Eq. (10) by following the steps outlined in Proposition 2.4:

$$s^{\alpha} U(x,s) - s^{\alpha-1}\, u(x,0) + 3\lambda \mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\} + 3\lambda \mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\} + \lambda \mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\} + \lambda U_{xxx}(x,s),\ s > 0. \tag{11}$$

where $U(x,s) = \mathcal{L}[u(x,\eta)]$ and $U_{xxx}(x,s) = \mathcal{L}[u_{xxx}(x,\eta)]$.

We obtain the next expression of Eq. (11) by dividing it by s^{α} and utilizing the initial conditions specified in Eq. (11):

$$U(x,s) = \frac{\zeta(x)}{s} - \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}\right) - \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right) - \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right) - \frac{\lambda}{s^{\alpha}} U_{xxx}(x,s),\ s > 0. \tag{12}$$

Expanding on the implications of Eq. (12), we obtain the following result:

$$U(x,s) = \sum_{j=0}^{\infty} \frac{\zeta_j(x)}{s^{1+\alpha j}}\ , s > 0. \tag{13}$$

Eq. (13) provides the expression for the kth-truncated series, which is given by:

$$U_k(x,s) = \frac{\zeta(x)}{s} + \sum_{j=1}^{k} \frac{\zeta_j(x)}{s^{1+\alpha j}}\ , s > 0. \tag{14}$$

We are able to delineate the primary LRPS methodologies, such as the LRF of Eq. (12), for the purpose of ascertaining the elusive value of the parameter $\zeta_j(x)$ that is exhibited as follows:

$$\mathrm{LRes}(\mathrm{x,s}) = U(x,s) - \frac{\zeta(x)}{s} + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right) + \frac{\lambda}{s^{\alpha}} U_{xxx}(x,s),\ s > 0. \tag{15}$$

In this context, the term kth-LRF refers to:

$$\mathrm{LRes}_k(\mathrm{x,s}) = \frac{\zeta(x)}{s} + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)^2\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)\right\}\right) + \frac{3\lambda}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{(k)xx}(x,s)\right)\right\}\right) + \frac{\lambda}{s^{\alpha}} U_{(k)xxx}(x,s),\ s > 0. \tag{16}$$

It is evident that for $s\ >\ 0$ and $k\ =\ 0, 1, 2, 3, \ldots$, $\mathrm{Lim}_{k\to\infty}\mathrm{LRes}_k(x,s) = \mathrm{LRes}(x,s)$, $\mathrm{LRes}(x,s) = 0$. Consequently, $\mathrm{Lim}_{s\to\infty}\left(s^k \mathrm{LRes}(x,s)\right) = 0$. Furthermore, it was demonstrated in reference [22.24] that...

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}(x,s)\right) = \mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(x,s)\right) = 0, k = 1,2,3,.. \tag{17}$$

The expression for $\mathrm{U}_1(x,s)$ given by $\mathrm{U}_1(x,s) = \frac{\zeta(x)}{s} + \frac{\zeta_1(x)}{s^{1+\alpha}}$, as shown in Eq. (16), represents...

$$\mathrm{LRes}_1(\mathrm{x},s)=\frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}}+\frac{3\lambda}{\mathrm{S}^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{\mathrm{s}}+\frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}}\right)\right)^2\right\}\right)+\frac{3\lambda}{\mathrm{S}^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{\mathrm{s}}+\frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}}\right)\right)^2\mathcal{L}^{-1}\left(\frac{\zeta'(\mathrm{x})}{\mathrm{S}}+\frac{\zeta_1'(\mathrm{x})}{\mathrm{S}^{\alpha+1}}\right)\right\}\right)+\frac{3\lambda}{\mathrm{S}^{\alpha}}\left(\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{\zeta(x)}{\mathrm{s}}+\frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}}\right)\mathcal{L}^{-1}\left(\frac{\zeta''(\mathrm{x})}{\mathrm{S}}+\frac{\zeta_1''(\mathrm{x})}{\mathrm{S}^{\alpha+1}}\right)\right\}\right)+\frac{\lambda}{\mathrm{S}^{\alpha}}\left(\frac{\zeta'''(\mathrm{x})}{\mathrm{S}}+\frac{\zeta_1'''(\mathrm{x})}{\mathrm{S}^{\alpha+1}}\right),\mathrm{s}>0. \qquad (18)$$

The application of the operator in Eq. (18) results in the following simplified expression:

$$\mathrm{LRes}_1(\mathrm{x},s)=\frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}}+\frac{3\lambda\,\zeta'^2(x)}{S^{\alpha+1}}+\frac{6\lambda\,\zeta'(x)\zeta_1'(x)}{S^{2\alpha+1}}+\frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}}+\frac{3\lambda\,\zeta'(x)\zeta^2(x)}{S^{\alpha+1}}+\frac{3\lambda\,\zeta_1'(x)\zeta^2(x)}{S^{2\alpha+1}}+\frac{6\lambda\,\zeta'(x)\zeta(x)\zeta_1(x)}{S^{2\alpha+1}}+\frac{6\lambda\,\zeta_1'(x)\zeta(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}}+\frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}}+\frac{3\lambda\,\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3S^{4\alpha+1}}+\frac{3\lambda\,\zeta(x)\zeta''(x)}{S^{\alpha+1}}+\frac{3\lambda\,\zeta(x)\zeta_1''(x)}{S^{2\alpha+1}}+\frac{3\lambda\,\zeta_1(x)\zeta''(x)\Gamma(\alpha+1)}{\Gamma(\alpha+1)S^{2\alpha+1}}+\frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{3\alpha+1}}+\frac{\lambda\,\zeta'''(x)}{S^{\alpha+1}}+\frac{\lambda\,\zeta_1'''(x)}{S^{2\alpha+1}},\mathrm{s}>0. \qquad (19)$$

By multiplying both sides of Eq. (19) by the term $s^{1+\alpha}$, we arrive at the following expression:

$$s^{1+\alpha}\mathrm{LRes}_1(\mathrm{x},\mathrm{s})=\zeta_1(x)+3\lambda\,\zeta'^2(x)+\frac{6\lambda\,\zeta'(x)\zeta_1'(x)}{S^{\alpha}}-\frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}}+3\lambda\,\zeta'(x)\zeta^2(x)+\frac{3\lambda\,\zeta_1'(x)\zeta^2(x)}{S^{\alpha}}+\frac{6\lambda\,\zeta'(x)\zeta(x)\zeta_1(x)}{S^{\alpha}}+\frac{6\lambda\,\zeta_1'(x)\zeta(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}}+\frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}}+\frac{3\lambda\,\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3S^{3\alpha}}+3\lambda\,\zeta(x)\zeta''(x)+\frac{3\lambda\,\zeta(x)\zeta_1''(x)}{S^{\alpha}}+\frac{3\lambda\,\zeta_1(x)\zeta''(x)\Gamma(\alpha+1)}{\Gamma(\alpha+1)S^{\alpha}}+\frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2S^{2\alpha}}+\lambda\,\zeta'''(x)+\frac{\lambda\,\zeta_1'''(x)}{S^{\alpha}},\mathrm{s}>0\,. \qquad (20)$$

Using the Eq. (17) assumption and the limit as s approaches infinity from both parts of Eq. (20), we can quickly determine the value of the function $\zeta_1(x)$ by solving the given formula for it.

$$0=s^{1+\alpha}\mathrm{LRes}_1(\mathrm{x},\mathrm{s})=\zeta_1(x)+3\lambda\zeta(x)\zeta''(x)+\lambda\zeta'''(x),\mathrm{s}>0\,. \qquad (21)$$

We can obtain the value of $\zeta_1(x)$ easily by applying the algebraic formula (21) to calculate it.

$$\zeta_1(x)=-\big(3\lambda\zeta(x)\zeta''(x)+\lambda\zeta'''(x)\big),\ \ s>0. \qquad (22)$$

To calculate the value of the next undetermined parameter$\zeta_2(x)$, the second-truncated series of Eq. (14), $\mathrm{U}_2(x,s)=\frac{\zeta(x)}{s}+\frac{\zeta_1(x)}{s^{1+\alpha}}+\frac{\zeta_2(x)}{s^{1+2\alpha}}$, is inserted into the second Linear Recurrence Formula (LRF).

$$\mathrm{LRes}_2(\mathrm{s}) = \frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}} + \frac{3\lambda}{\mathrm{s}^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{\mathrm{s}} + \frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}} + \frac{\zeta_2(x)}{s^{1+2\alpha}}\right)\right)^2\right\}\right) + \frac{3\lambda}{\mathrm{s}^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(\frac{\zeta(x)}{\mathrm{s}} + \frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}} + \frac{\zeta_2(x)}{s^{1+2\alpha}}\right)\right)^2 \mathcal{L}^{-1}\left(\frac{\zeta'(\mathrm{x})}{\mathrm{s}} + \frac{\zeta_1'(\mathrm{x})}{\mathrm{s}^{\alpha+1}} + \frac{\zeta_2'(x)}{s^{1+2\alpha}}\right)\right\}\right) + \frac{3\lambda}{\mathrm{s}^{\alpha}}\left(\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{\zeta(x)}{\mathrm{s}} + \frac{\boldsymbol{\zeta_1}(x)}{\boldsymbol{s^{1+\alpha}}} + \frac{\zeta_2(x)}{s^{1+2\alpha}}\right)\mathcal{L}^{-1}\left(\frac{\zeta''(\mathrm{x})}{\mathrm{s}} + \frac{\zeta_1''(\mathrm{x})}{\mathrm{s}^{\alpha+1}} + \frac{\zeta_2''(x)}{s^{1+2\alpha}}\right)\right\}\right) + -\frac{\lambda}{\mathrm{s}^{\alpha}}\left(\frac{\zeta'''(\mathrm{x})}{\mathrm{s}} + \frac{\zeta_1'''(\mathrm{x})}{\mathrm{s}^{\alpha+1}} + \frac{\zeta_2'''(x)}{s^{1+2\alpha}}\right), \mathrm{s} > 0. \quad (23)$$

The operator in Eq. (18) can be applied to yield the following simplified expression.

$$\mathrm{LRes}_2(\mathrm{x}, s) = \frac{\zeta_2(x)}{S^{2\alpha+1}} + \frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{3\alpha+1}} + \frac{3\lambda\zeta_2'^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{5\alpha+1}} + \frac{6\lambda\zeta'(x)\zeta_1'(x)}{S^{2\alpha+1}} + \frac{6\lambda\zeta'(x)\zeta_2'(x)}{S^{3\alpha+1}} + \frac{6\lambda\zeta_1'(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{4\alpha+1}} + \frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{3\alpha+1}} + \frac{3\lambda\,\zeta'(x)\zeta_2^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{5\alpha+1}} + \frac{6\lambda\zeta'(x)\zeta(x)\zeta_1(x)}{S^{2\alpha+1}} + \frac{6\lambda\zeta'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)S^{4\alpha+1}} + \frac{6\lambda\zeta'(x)\zeta_1(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{4\alpha+1}} + \frac{3\lambda\zeta_1'(x)\zeta^2(x)}{S^{2\alpha+1}} + \frac{3\lambda\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3 S^{4\alpha+1}} + \frac{3\lambda\zeta_1'(x)\zeta_2^2(x)\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2 S^{6\alpha+1}} + \frac{6\lambda\zeta_1'(x)\,\zeta(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{3\alpha+1}} + \frac{6\lambda\zeta_1'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{4\alpha+1}} + \frac{6\lambda\zeta_1'(x)\zeta_1(x)\zeta_2(x)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)S^{5\alpha+1}} + \frac{3\lambda\zeta^2(x)\zeta_2'(x)}{S^{3\alpha+1}} + \frac{3\lambda\zeta_1^2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)^2 S^{5\alpha+1}} + \frac{3\lambda\,\zeta_2^2(x)\zeta_2'(x)\Gamma(6\alpha+1)}{\Gamma(2\alpha+1)^3 S^{7\alpha+1}} + \frac{6\lambda\,\zeta(x)\zeta_1(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)S^{4\alpha+1}} + \frac{6\lambda\,\zeta(x)\zeta_2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{5\alpha+1}} + \frac{6\lambda\zeta_1(x)\zeta_2(x)\zeta_2'(x)\,\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2 S^{6\alpha+1}} + \frac{3\lambda\zeta(x)\zeta_1''(x)}{S^{2\alpha+1}} + \frac{3\lambda\zeta(x)\zeta_2''(x)}{S^{3\alpha+1}} + \frac{3\lambda\zeta_1(x)\zeta''(x)}{S^{2\alpha+1}} + \frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{3\alpha+1}} + \frac{3\lambda\zeta_1(x)\zeta_2''(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{4\alpha+1}} + \frac{3\lambda\,\zeta_2(x)\zeta''(x)}{S^{3\alpha+1}} + \frac{3\lambda\,\zeta_2(x)\zeta_1''(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)S^{4\alpha+1}} + \frac{3\lambda\,\zeta_2(x)\zeta''{}_2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{5\alpha+1}} + \frac{\lambda\,\zeta_1'''(x)}{S^{2\alpha+1}} + \frac{\lambda\,\zeta_2'''(x)}{S^{3\alpha+1}}, \mathrm{s} > 0. \quad (24)$$

Multiplying Eq. (24) by $s^{1+2\alpha}$ results in the following products.

$$s^{1+2\alpha}\mathrm{LRes}_2(\mathrm{x},\mathrm{s}) = \zeta_2(x) + \frac{3\lambda\,\zeta_1'^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{\alpha}} + \frac{3\lambda\zeta_2'^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{3\alpha}} + 6\lambda\zeta'(x)\zeta_1'(x) + \frac{6\lambda\zeta'(x)\zeta_2'(x)}{S^{\alpha}} + \frac{6\lambda\zeta_1'(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{2\alpha}} + \frac{3\lambda\,\zeta'(x)\zeta_1^2(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{\alpha}} + \frac{3\lambda\,\zeta'(x)\zeta_2^2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{3\alpha}} + 6\lambda\zeta'(x)\zeta(x)\zeta_1(x) + \frac{6\lambda\zeta'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)S^{2\alpha}} + \frac{6\lambda\zeta'(x)\zeta_1(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{2\alpha}} + 3\lambda\zeta_1'(x)\zeta^2(x) + \frac{3\lambda\zeta_1'(x)\zeta_1^2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3 S^{2\alpha}} + \frac{3\lambda\zeta_1'(x)\zeta_2^2(x)\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2 S^{4\alpha}} + \frac{6\lambda\zeta_1'(x)\,\zeta(x)\zeta_1(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{\alpha}} + \frac{6\lambda\zeta_1'(x)\zeta(x)\zeta_2(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{2\alpha}} + \frac{6\lambda\zeta_1'(x)\zeta_1(x)\zeta_2(x)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)S^{3\alpha}} + \frac{3\lambda\zeta^2(x)\zeta_2'(x)}{S^{\alpha}} + \frac{3\lambda\zeta_1^2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)^2 S^{3\alpha}} + \frac{3\lambda\,\zeta_2^2(x)\zeta_2'(x)\Gamma(6\alpha+1)}{\Gamma(2\alpha+1)^3 S^{5\alpha}} + \frac{6\lambda\,\zeta(x)\zeta_1(x)\zeta_2'(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)S^{2\alpha}} + \frac{6\lambda\,\zeta(x)\zeta_2(x)\zeta_2'(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{3\alpha}} + \frac{6\lambda\zeta_1(x)\zeta_2(x)\zeta_2'(x)\,\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2 S^{4\alpha}} + 3\lambda\zeta(x)\zeta_1''(x) + \frac{3\lambda\zeta(x)\zeta_2''(x)}{S^{\alpha}} + 3\lambda\zeta_1(x)\zeta''(x) + \frac{3\lambda\,\zeta_1(x)\zeta_1''(x)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 S^{\alpha}} + \frac{3\lambda\zeta_1(x)\zeta_2''(x)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)S^{2\alpha}} + \frac{3\lambda\,\zeta_2(x)\zeta''(x)}{S^{\alpha}} + \frac{3\lambda\,\zeta_2(x)\zeta_1''(x)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)S^{2\alpha}} + \frac{3\lambda\,\zeta_2(x)\zeta''{}_2(x)\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2 S^{3\alpha}} + \lambda\,\zeta_1'''(x) + \frac{\lambda\,\zeta_2'''(x)}{S^{\alpha}}, \mathrm{s} > 0. \quad (25)$$

To obtain the following formula, take the limit as s approaches infinity for both parts of Eq. (25), and then apply Eq. (17).

$$0 = \zeta_2(x) + 3\lambda\zeta_1(x)\zeta''(x) + 3\lambda\zeta(x)\zeta_1''(x) + \lambda\,\zeta_1'''(x), \mathrm{s} > 0. \quad (26)$$

Solving the resulting algebraic Equation for $\zeta_2(x)$ yields the following expression.

$$\zeta_2(x) = -(3\lambda\zeta_1(x)\zeta''(x) + 3\lambda\zeta(x)\zeta_1''(x) + \lambda\,\zeta_1'''(x)), s > 0. \tag{27}$$

Similar to previous stages, we insert the third-truncated series of Eq. (17), $U_3(x,s) = \frac{\zeta(x)}{s} + \frac{\zeta_1(x)}{s^{1+\alpha}} + \frac{\zeta_2(x)}{s^{1+2\alpha}} + \frac{\zeta_3(x)}{s^{1+3\alpha}}$, into the third Linear Recurrence Formula (LRF) to obtain the value of the next undetermined parameter, $\zeta_3(x)$.

$$\zeta_3(x) = -(3\lambda\zeta_2(x)\zeta''(x) + \frac{3\lambda\Gamma(1+2\alpha)\zeta_1(x)\zeta_1{}''(x)}{\Gamma(1+\alpha)^2} + 3\lambda\zeta(x)\zeta_2{}''(x) + \lambda\zeta_2{}'''(x)), s > 0. \tag{28}$$

Consequently, we can express the solution of Eq. (14) as an infinite series, as follows.

$$U(x,s) = \frac{\zeta(x)}{s} - \frac{\left(3\lambda\zeta(x)\zeta''(x)+\lambda\zeta'''(x)\right)}{s^{1+\alpha}} - \frac{\left(3\lambda\zeta_1(x)\zeta''(x)+3\lambda\zeta(x)\zeta_1{}''(x)+\lambda\zeta_1{}'''(x)\right)}{s^{1+2\alpha}} - \frac{(3\lambda\zeta_2(x)\zeta''(x)+\frac{3\lambda\Gamma(1+2\alpha)\zeta_1(x)\zeta_1{}''(x)}{\Gamma(1+\alpha)^2}+3\lambda\zeta(x)\zeta_2{}''(x)+\lambda\zeta_2{}'''(x))}{s^{1+3\alpha}} - \ldots \tag{29}$$

The solution of Eqs. (1) and (9) using the LRPS method can be obtained by applying the inverse Laplace transform of Eq. (29) in the specified simple form.

$$u(x,t) = \zeta(x) - \frac{\left(3\lambda\zeta(x)\zeta''(x)+\lambda\zeta'''(x)\right)}{\Gamma(1+\alpha)}t^{\alpha} - \frac{\left(3\lambda\zeta_1(x)\zeta''(x)+3\lambda\zeta(x)\zeta_1{}''(x)+\lambda\zeta_1{}'''(x)\right)}{\Gamma(1+2\alpha)}t^{2\alpha} - \frac{(3\lambda\zeta_2(x)\zeta''(x)+\frac{3\lambda\Gamma(1+2\alpha)\zeta_1(x)\zeta_1{}''(x)}{\Gamma(1+\alpha)^2}+3\lambda\zeta(x)\zeta_2{}''(x)+\lambda\zeta_2{}'''(x))}{\Gamma(1+3\alpha)}t^{3\alpha} - \ldots \tag{30}$$

4 Numerical Issues

In this section, we examine the significance of the LRPSM in finding the solution to the FSTOF formula.

Problem 4.1. Consider the fractional equation presented below.

$$D_{t*}^{\alpha}u(x,t) + 3u_x^2(x,t) + 3u^2(x,t)u_x(x,t) + 3u(x,t)u_{xx}(x,t) + u_{xxx}(x,t) = 0$$
$$t > 0, 0 < \alpha \leq 1, \tag{31}$$

And initial condition as:

$$u(\mathrm{x},0) = \frac{2\,k\,(\,tanh\,(k\,x)+w)}{w\,tanh(k\,x)+1}, \qquad k\,, w \in C \tag{32}$$

Applying the Laplace transform to Eq. (32) and then using it in Eq. (31) yields the following expression.

$$U(x,s) = \frac{2\,k\,(\,tanh\,(k\,x)+w)}{s\,(\,w\,tanh(k\,x)+1)} + \frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_x(x,s)\right)^2\right)\right\}\right) + \frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right) + \frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right) + \frac{1}{s^{\alpha}}U_{xxx}(x,s),\ s > 0. \tag{33}$$

It is asserted that the kth-truncated series can be expressed as follows.

$$U_k(x,s) = \frac{2\,k\,(\,tanh\,(k\,x)+w)}{s\,(\,w\,tanh(k\,x)+1)} + \sum_{j=1}^{k}\frac{\zeta_j(x)}{s^{1+\alpha j}}\,, s > 0. \tag{34}$$

Thus, the kth Linear Recurrence Formulas (LRFs) are given by the following expressions.

$$\mathrm{LRes}_k(s) = \frac{2\,k\,(\,tanh\,(k\,x)+w)}{s(w\,tanh(k\,x)+1)} - \frac{3}{s^\alpha}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)^2\right)\right\}\right) - \frac{3}{s^\alpha}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)\right\}\right)\ \frac{3}{s^\alpha}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{(k)xx}(x,s)\right)\right\}\right) - \frac{1}{s^\alpha}U_{(k)xxx}(x,s),\ s>0. \tag{35}$$

The kth-truncated series of Eq. (34) is inserted into the kth Linear Recurrence Formula (LRF) of Eq. (35) to obtain $\zeta_j(x)$. Multiplying the resulting formula by $s^{1+\alpha j}$, allows us to determine the relationship.

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(s)\right) = 0, k = 1,2,3,..$$

So, several values include:

$$\zeta_1(x) = -\frac{4ak^4(w^2-1)\mathrm{sech}^4(kx)(2(w^2+1)\cosh(2kx)+4w\sinh(2kx)+w^2-1)}{(\mathrm{wtanh}(kx)+1)^4},$$

$$\zeta_2(x) = \frac{1}{(w\tanh(kx)+1)^7}16a^2k^7(w^2-1)\mathrm{sech}^7(kx)(w^5\cosh(5kx) - w^4\sinh(kx) + 27w^4\sinh(3kx) + 5w^4\sinh(5kx) + 10w^3\cosh(5kx) + 2w^2\sinh(kx) - 18w^2\sinh(3kx) + 10w^2\sinh(5kx) - (w^2-1)^2w\cosh(kx) + 9(w^4+2w^2-3)w\cosh(3kx) + 5w\cosh(5kx) - \sinh(kx) - 9\sinh(3kx) + \sinh(5kx)),$$

$$\zeta_3(x) = -\frac{1}{\Gamma(\alpha+1)^2(w\sinh(kx)+\cosh(kx))^{10}}32a^3k^{10}(w^2-1)(3(w^2-1)\Gamma(2\alpha+1)(w^6\cosh(6kx) + 90w^5\sinh(2kx) + 60w^5\sinh(4kx) + 6w^5\sinh(6kx) + 15w^4\cosh(6kx) - 180w^3\sinh(2kx) + 20w^3\sinh(6kx) + 15w^2\cosh(6kx) + 45(w^2-1)^2(w^2+1)\cosh(2kx) + 15(w^6+5w^4-5w^2-1)\cosh(4kx) + 90w\sinh(2kx) - 60w\sinh(4kx) + 6w\sinh(6kx) + \cosh(6kx) + 29w^6 - 87w^4 + 87w^2 - 29) + \Gamma(\alpha+1)^2(28w^8\cosh(6kx) + w^8\cosh(8kx) - 940w^7\sinh(2kx) - 320w^7\sinh(4kx) + 168w^7\sinh(6kx) + 8w^7\sinh(8kx) + 392w^6\cosh(6kx) + 28w^6\cosh(8kx) + 2820w^5\sinh(2kx) + 320w^5\sinh(4kx) + 392w^5\sinh(6kx) + 56w^5\sinh(8kx) + 70w^4\cosh(8kx) - 2820w^3\sinh(2kx) + 320w^3\sinh(4kx) - 392w^3\sinh(6kx) + 56w^3\sinh(8kx) - 392w^2\cosh(6kx) + 28w^2\cosh(8kx) - 470(w^2-1)^3(w^2+1)\cosh(2kx) - 80(w^2-1)^2(w^4+6w^2+1)\cosh(4kx) + 940w\sinh(2kx) - 320w\sinh(4kx) - 168w\sinh(6kx) + 8w\sinh(8kx) - 28\cosh(6kx) + \cosh(8kx) - 181w^8 + 724w^6 - 1086w^4 + 724w^2 - 181)),$$

Therefore, we can express the solution of Eq. (34) as an infinite series, as follows.

$U(x,s) =$

$$\frac{2k(tanh\,(kx)+w)}{s(wtanh\,(kx)+1)}+$$
$$\frac{1}{s^{1+\alpha}}\left(-\frac{4ak^4(w^2-1)sech^4(kx)\left(2\left(w^2+1\right)cosh(2kx)+4w\,sinh(2kx)+w^2-1\right)}{(wtanh(kx)+1)^4}\right)+$$
$$\frac{1}{s^{2\alpha+1}}\left(\frac{1}{(\mathrm{wtanh}(kx)+1)^7}16a^2k^7(w^2-1)\mathrm{sech}^7(kx)(w^5\cosh(5kx)-w^4\sinh(kx)+27w^4\sinh(3kx)+5w^4\sinh(5kx)+10w^3\cosh(5kx)+2w^2\sinh(kx)-18w^2\sinh(3kx)+10w^2\sinh(5kx)-(w^2-1)^2w\cosh(kx)+9(w^4+2w^2-3)w\cosh(3kx)+5w\cosh(5kx)-\sinh(kx)-9\sinh(3kx)+\sinh(5kx))\right)+$$
$$\frac{1}{s^{3\alpha+1}}\left(-\frac{1}{\Gamma(\alpha+1)^2(w\sinh(kx)+\cosh(kx))^{10}}32a^3k^{10}(w^2-1)(3(w^2-1)\Gamma(2\alpha+1)(w^6\cosh(6kx)+90w^5\sinh(2kx)+60w^5\sinh(4kx)+6w^5\sinh(6kx)+15w^4\cosh(6kx)-180w^3\sinh(2kx)+20w^3\sinh(6kx)+15w^2\cosh(6kx)+45(w^2-1)^2(w^2+1)\cosh(2kx)+15(w^6+5w^4-5w^2-1)\cosh(4kx)+90w\sinh(2kx)-60w\sinh(4kx)+6w\sinh(6kx)+\cosh(6kx)+29w^6-87w^4+87w^2-29)+\Gamma(\alpha+1)^2(28w^8\cosh(6kx)+w^8\cosh(8kx)-940w^7\sinh(2kx)-320w^7\sinh(4kx)+168w^7\sinh(6kx)+8w^7\sinh(8kx)+392w^6\cosh(6kx)+28w^6\cosh(8kx)+2820w^5\sinh(2kx)+320w^5\sinh(4kx)+392w^5\sinh(6kx)+56w^5\sinh(8kx)+70w^4\cosh(8kx)-2820w^3\sinh(2kx)+320w^3\sinh(4kx)-392w^3\sinh(6kx)+56w^3\sinh(8kx)-392w^2\cosh(6kx)+28w^2\cosh(8kx)-470(w^2-1)^3(w^2+1)\cosh(2kx)-80(w^2-1)^2(w^4+6w^2+1)\cosh(4kx)+940w\sinh(2kx)-320w\sinh(4kx)-168w\sinh(6kx)+8w\sinh(8kx)-28\cosh(6kx)+\cosh(8kx)-181w^8+724w^6-1086w^4+724w^2-181))\right)+.. \quad (36)$$

If we perform the inverse Laplace transform, the resulting expression is as follows.

$$u(x,t)=\frac{2k(\tanh(kx)+w)}{w\tanh(kx)+1}-\frac{4ak^4(w^2-1)t^{\alpha}\mathrm{sech}^4(kx)(2(w^2+1)\cosh(2kx)+4w\sinh(2kx)+w^2-1)}{\Gamma(\alpha+1)(w\tanh(kx)+1)^4}+\frac{1}{\Gamma(2\alpha+1)(w\tanh(kx)+1)^7}16a^2k^7(w^2-1)t^{2\alpha}\mathrm{sech}^7(kx)(w^5\cosh(5kx)-w^4\sinh(kx)+27w^4\sinh(3kx)+5w^4\sinh(5kx)+10w^3\cosh(5kx)+2w^2\sinh(kx)-18w^2\sinh(3kx)+10w^2\sinh(5kx)-(w^2-1)^2w\cosh(kx)+9(w^4+2w^2-3)w\cosh(3kx)+5w\cosh(5kx)-\sinh(kx)-9\sinh(3kx)+\sinh(5kx))-\frac{1}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)(w\sinh(kx)+\cosh(kx))^{10}}32a^3k^{10}(w^2-1)t^{3\alpha}(3(w^2-1)\Gamma(2\alpha+1)(w^6\cosh(6kx)+90w^5\sinh(2kx)+60w^5\sinh(4kx)+6w^5\sinh(6kx)+15w^4\cosh(6kx)-180w^3\sinh(2kx)+20w^3\sinh(6kx)+15w^2\cosh(6kx)+45(w^2-1)^2(w^2+1)\cosh(2kx)+15(w^6+5w^4-5w^2-1)\cosh(4kx)+90w\sinh(2kx)-60w\sinh(4kx)+6w\sinh(6kx)+\cosh(6kx)+29w^6-87w^4+87w^2-29)+\Gamma(\alpha+1)^2(28w^8\cosh(6kx)+w^8\cosh(8kx)-940w^7\sinh(2kx)-320w^7\sinh(4kx)+168w^7\sinh(6kx)+8w^7\sinh(8kx)+392w^6\cosh(6kx)+28w^6\cosh(8kx)+2820w^5\sinh(2kx)+320w^5\sinh(4kx)+392w^5\sinh(6kx)+56w^5\sinh(8kx)+70w^4\cosh(8kx)-2820w^3\sinh(2kx)+320w^3\sinh(4kx)-392w^3\sinh(6kx)+56w^3\sinh(8kx)-392w^2\cosh(6kx)+28w^2\cosh(8kx)-470(w^2-1)^3(w^2+1)\cosh(2kx)-80(w^2-1)^2(w^4+6w^2+1)\cosh(4kx)+940w\sinh(2kx)-320w\sinh(4kx)-168w\sinh(6kx)+8w\sinh(8kx)-28\cosh(6kx)+\cosh(8kx)-181w^8+724w^6-1086w^4+724w^2-181))+.. \quad (37)$$

Due to the unpredictable nature of the coefficients in the series solution of Eq. (37), it is not possible to obtain an exact solution. Hence, we evaluate the results using the residual and relative errors, which are defined as follows, respectively:

$$\text{Res. Err}\ (x,t) = |\mathcal{L}^{-1}[LRes_5(x,s)]| = |D_t^{\alpha}u(x,t) + 3u_x^2(x,t) + 3u^2(x,t)u_x(x,t) + 3u(x,t)u_{xx}(x,t) + u_{xxx}(x,t)| \quad (38)$$

$$\text{Rel. Err}\ (x,t) = \left|\frac{u_5(x,t)-u_3(x,t)}{u_5(x,t)}\right|. \quad (39)$$

Song et al. [25] have previously solved equation (1) using the Variation iteration approach, Adomian decomposition, and homotopy perturbation method. Additionally, the Residual Power Series Technique [26] was employed. However, in this study, we solve equation (1) using the LRPSM.

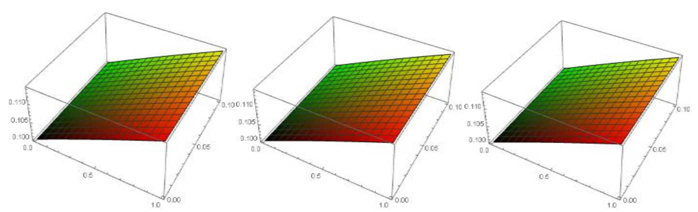
Table 4.1 displays the numerical solutions to the problem, along with the residual and relative errors for various values of \alpha within the range $(0,\infty)\times[0,1]$, and the fifth approximate result. The results indicate that the LRPS approach is a reliable numerical method for solving a nonlinear FSTOF.

Figure 4.1a, b, and c illustrate the graphs of the 5th approximate solutions to equations (31) and (32) over the range $(0,\infty)\times[0,1]$. The graphs indicate that the solutions to the IVPs (31) and (32) are strictly decreasing throughout the region.

Table 1. Assessment among the Exact solution, VIM, ADM, and HPM and LRPS solution with parameters $w = 0.5$, $k = a = \alpha = 1$ and $t = 0.001$

x	Exact Solution	VIM	ADM	HPM	LRPS
0	0.938808808	0.938798380	0.938800000	0.938800000	0.938808808
1	1.813631681	1.813642383	1.813642415	1.813642415	1.813631681
2	1.973719022	1.973721044	1.973721044	1.973721044	1.973719022
3	1.996422935	1.996423221	1.996423221	1.996423221	1.996422935
4	1.999515521	1.999515561	1.999515561	1.999515561	1.999515522
5	1.999934426	1.999934431	1.999934431	1.999934431	1.999934428
6	1.999991125	1.999991127	1.999991127	1.999991127	1.999991129
7	1.999998799	1.999998799	1.999998799	1.999998799	1.999998799

Above table shows that numerical values of the obtained solution are also very nearer to the exact solution.



(a) (b) (c)

Fig. 1. The graphs of Eqs. (31) at various values 0f α: (a) $\alpha = 1$, (b) $\alpha = 0.9, (c)\alpha =$

0.8.

Problem 4.2: Consider the fractional equation presented below.

$$D_{t*}^{\alpha}u(x,t) + 3u^2(x,t)u_x(x,t) + 3u(x,t)u_{xx}(x,t) = 0, t > 0, 0 < \alpha \leq 1, \quad (40)$$

And initial condition as:

$$u(\mathrm{x},0) = \pi \cosh(5x) \quad (41)$$

Applying the Laplace transform to equation (41) and then using it in equation (40) yields the following expression.

$$U(x,s) = \frac{\pi \cosh(5x)}{\mathrm{s}} + \frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_x(x,s)\right)\right\}\right) +$$

$$\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{xx}(x,s)\right)\right\}\right), \ s > 0. \quad (42)$$

It is claimed that the kth-truncated series can be expressed using the following formula.

$$U_k(x,s) = \frac{\pi \cosh(5x)}{s} + \sum_{j=1}^{k} \frac{\zeta_j(x)}{s^{1+\alpha j}}, s > 0. \quad (43)$$

Thus, the kth Linear Recurrence Formulas (LRFs) are given by the following expressions.

$$\mathrm{LRes}_k(\mathrm{s}) = \frac{\pi \cosh(5x)}{\mathrm{s}} - \frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)^2\right)\mathcal{L}^{-1}\left(U_{(k)x}(x,s)\right)\right\}\right) -$$

$$\frac{3}{s^{\alpha}}\left(\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left(U_k(x,s)\right)\right)\mathcal{L}^{-1}\left(U_{(k)xx}(x,s)\right)\right\}\right), \ s > 0. \quad (44)$$

The kth-truncated series of equation (43) is inserted into the kth Linear Recurrence Formula (LRF) of equation (44) to obtain $\zeta_3(x)$. Multiplying the resulting formula by $s^{1+\alpha j}$ allows us to determine the relationship.

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(s)\right) = 0, k = 1,2,3, ..$$

So, several values include:

$$\zeta_1(x) = -75\pi^2 \cosh^2(5x),$$

$$\zeta_2(x) = \frac{5625}{4}\pi^3(7\cosh(5x) + 5\cosh(15x))$$

$$\zeta_3(x) = -\frac{421875\pi^4\cosh^2(5x)(\Gamma(\alpha+1)^2(25\cosh(10x) - 9) + 2\Gamma(2\alpha+1)\cosh(10x))}{\Gamma(\alpha+1)^2},$$

Therefore, we can express the solution of equation (43) as an infinite series, as follows.

$$U(x,s) = \frac{\pi \cosh(5x)}{s} + \frac{(-75\pi^2\cosh^2(5x))}{s^{1+\alpha}} + \frac{1}{s^{2\alpha+1}}\left(\frac{5625}{4}\pi^3(7\cosh(5x) + 5\cosh(15x))\right) +$$

$$\frac{1}{s^{3\alpha+1}}\left(-\frac{421875\pi^4\cosh^2(5x)(\Gamma(\alpha+1)^2(25\cosh(10x)-9)+2\Gamma(2\alpha+1)\cosh(10x))}{\Gamma(\alpha+1)^2}\right) + \cdots \quad (45)$$

If we perform the inverse Laplace transform, the resulting expression is as follows.

$$u(x,t) = \pi \cosh(5x) - \frac{75\pi^2 t^{\alpha} \cosh^2(5x)}{\Gamma(\alpha+1)} + \frac{5625\, \pi^3 t^{2\alpha} \cosh(5x)}{2\Gamma(2\alpha+1)} + \frac{28125\, \pi^3 t^{2\alpha} \cosh(5x) \cosh(10x)}{2\Gamma(2\alpha+1)} - \frac{2953125\pi^4 t^{3\alpha} \cosh^2(5x)}{2\Gamma(3\alpha+1)} - \frac{10546875\pi^4 t^{3\alpha} \cosh(5x)\cosh(15x)}{2\Gamma(3\alpha+1)} - \frac{421875\pi^4 \Gamma(2\alpha+1) t^{3\alpha} \sinh(20x) \coth(5x)}{2\Gamma(\alpha+1)^2 \Gamma(3\alpha+1)} + \ldots \quad (46)$$

Due to the unpredictable nature of the coefficients in the series solution of equation (46), it is not possible to obtain an exact solution. Hence, we evaluate the results using the residual and relative errors, which are defined as follows, respectively:

Tables 4.2, 4.3, and 4.4 display the numerical solutions to the problem, along with the residual and relative errors for various values of α within the range $(0,\infty) \times [0,1]$, and the 5th approximate result. The results indicate that the LRPS approach is a reliable numerical method for solving a nonlinear FSTOF.

Figure 4.2a, b, c, d, and e display the graphs of the 5th approximate solutions to equations (40) and (41) over the range $(0,\infty) \times [0,1]$. The graphs demonstrate that the solutions to the IVPs (40) and (41) are strictly decreasing throughout the region.

Table 2. Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 1$.

x	t	$\boldsymbol{u_5(x,t)}$ − approximation	$\boldsymbol{Res.Err.(x,t)}$	Rel. $\boldsymbol{Err\ (x,t)}$
0.01	0.000001	3.1447786530713	3.1447786530710635	7.51263×10^{-14}
	0.000002	3.1440373687876986	3.1440373687839247	$1.20033 * 10^{-12}$
	0.000003	3.1432966089257905	3.1432966089067045	6.07199×10^{-12}
	0.000004	3.142556372669747	3.1425563726094805	1.91775×10^{-11}
0.03	0.000001	3.176245162469971	3.1762451624692853	2.15876×10^{-13}
	0.000002	3.1754889845399843	3.1754889845290157	3.45413×10^{-12}
	0.000003	3.1747333539725484	3.1747333539170786	1.74723×10^{-11}
	0.000004	3.1739782699128454	3.1739782697376997	5.51818×10^{-11}
0.05	0.000001	3.2394926597611433	3.239492659759883	3.8905×10^{-13}
	0.000002	3.2387060981693567	3.2387060981492004	6.22357×10^{-12}
	0.000003	3.237920131631696	3.2379201315297563	3.14831×10^{-11}
	0.000004	3.237134759187277	3.2371347588654102	9.94296×10^{-11}

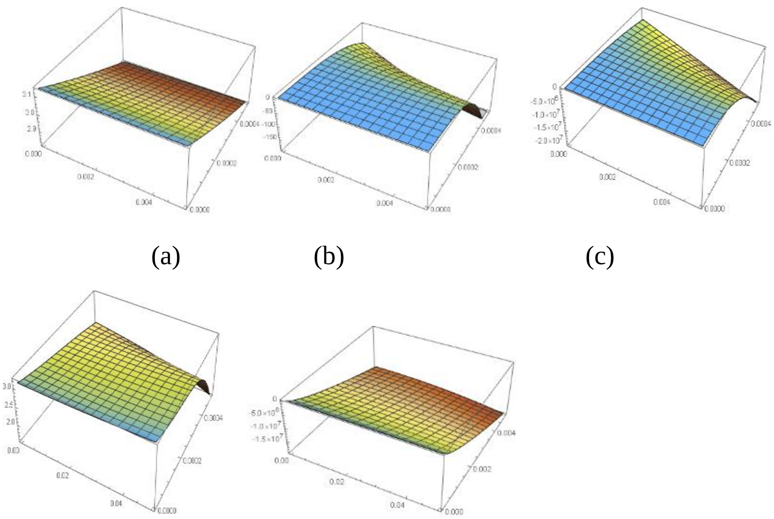
Table 3. Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 0.75$

x	t	$u_5(x,t)-$ approximation	$Res.Err.(x,t)$	Rel. $Err\ (x,t)$
0.01	0.000001	3.1203742229907303	3.120373378816346	2.70536×10^{-7}
	0.000002	3.103658233056388	3.1036517323911657	2.09452×10^{-6}
	0.000003	3.0892786400996304	3.0892574481304345	6.85989×10^{-6}
	0.000004	3.0762937969403112	3.0762451919325455	1.58001×10^{-5}
0.03	0.000001	3.1513602072951548	3.151357711683688	7.91916×10^{-7}
	0.000002	3.1343338076067475	3.134314603022589	6.1272×10^{-6}
	0.000003	3.1197113278975652	3.1196487620206126	2.00554×10^{-5}
	0.000004	3.1065380616574227	3.1063946544227803	4.61652×10^{-5}
0.05	0.000001	3.2136267165073926	3.213622122424383	1.42957×10^{-6}
	0.000002	3.195959471545515	3.195924178997689	1.1043×10^{-5}
	0.000003	3.180820240678176	3.1807054485145523	3.60902×10^{-5}
	0.000004	3.167221783571589	3.1669590880932823	8.29488×10^{-5}

Table 4. Numerical comparisons between the 5th-approximation of $u_5(x,t)$ and the residual error of $u(x,t)$ at $\alpha = 0.90$.

x	t	$u_5(x,t)-$ approximation	$Res.Err.(x,t)$	Rel. $Err\ (x,t)$
0.01	0.000001	3.142453736729295	3.142453736626076	3.28467×10^{-11}
	0.000002	3.1398057312452234	3.139805729998697	3.97008×10^{-10}
	0.000003	3.1372998473021236	3.137299841957196	1.70367×10^{-9}
	0.000004	3.13488402680838	3.1348840118086647	4.78478×10^{-9}

0.03	0.000001	3.173873587258227	3.1738735869558683	9.52648×10^{-11}
	0.000002	3.1711726029754224	3.1711725993242035	1.15138×10^{-9}
	0.000003	3.168616747998005	3.168616732342718	4.94073×10^{-9}
	0.000004	3.1661529081632724	3.166152864230786	1.38757×10^{-8}
0.05	0.000001	3.237025902207323	3.2370259016507537	1.71939×10^{-10}
	0.000002	3.234216853471912	3.234216846751956	2.07777×10^{-9}
	0.000003	3.2315590761982347	3.2315590473896783	8.91475×10^{-9}
	0.000004	3.228997296750303	3.228997215918956	2.5033×10^{-8}



(a) (b) (c)

(d) (e)

FIG. 2. The graphs of Eqs. (40) at various values of α: (a) $\alpha = 1$, (b) $\alpha = 0.75$, (c) $\alpha = 0.90$, (d) $\alpha = 0.50$, (e) $\alpha = 0.25$.

5 Conclusion

In this study, LRPSM was successfully used to obtain the solution of the fractional Sharma-Tasso-Oliver equation. Based on the results obtained, it was found that LRPSM is very effective and accurate in solving differential equations of fractional order, such as the Sharma-Tasso-Oliver problem, and it was compared with several methods such as VIM, HPM, RPS, where it was concluded that the LRPS approach is a powerful and advanced method for finding Both approximate and analytical solutions to various partial mathematical models that arise in different scientific fields.

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