On Third-Order Differential Subordinations and Superordinations Results for Univalent Analytic Functions Defined by a New Operator

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Abstract: In the current paper, we obtain sandwich theorems for univalent functions by using some of the finding of Third-order differential subordination and superordination for univalent functions defined by a new operator $Ʀ^n f(z)$.

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1-Introduction and Definitions

Let $\hat{W} = \hat{W}(U)$ be the class functions which are analytic in the open unit disk $U = \{z \in C: |z| < 1\}$. For $n \in N$ and $a \in C$ .Let $\hat{W}[a,n]$ be the subclass of $\hat{W}$ and

$$\hat{W}[a,n] = \{f \in \hat{W}: f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \cdots.\}, (a \in C)$$

Let A denote the subclass of $\hat{W}$ of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0, a \in C), \qquad (1.1)$$

Suppose f and g are analytic functions in $\hat{W}$. We say that f is subordinate to g, or g is superordinate to f in U, and write $f \prec g$, or $f(z) \prec g(z)$, if there exists a Schwarz function $\dot{w}$ in U , which with $\dot{w}(0) = 0$, and $|\dot{w}(z)| < 1$, $(z \in U)$,where $f(z) = g(\dot{w}(z))$. Furthermore, if g is univalent in U, we have(see[8]), $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ $z \in U$.

<u>Definition(1.1)</u>[1]: Let $\acute{\emptyset}: \mathbb{C}^4 \times U \to \mathbb{C}$ and suppose that the function $Ķ(z)$ be univalent in U. If $ᵽ(z)$ is analytic in U and satisfies the following:

$$\acute{\emptyset}(ᵽ(z), z ᵽ'(z), z^2 ᵽ''(z), z^3 ᵽ'''(z); z) \prec Ķ(z), \qquad (1.2)$$

then $ᵽ(z)$ is called a solution of the differential subordination (1,2). Furthermore, given univalent function $ᵽ(z)$ is called a dominant of the solution of the differential subordination (1.2) if , $ᵽ(z) \prec q(z)$ for all $ᵽ(z)$ satisfying (1.2). A dominant $\check{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$ of (1.2) is said to be the bsst dominant.

<u>Definition (1.2)</u>[3,24]) : For $f \in A$,we define An operator $Ʀ^n: A \to A$

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, (n \in N). \quad (1.3)$$

When $\delta = 1$, we get Salagean's differential operator (see[3]),
and

$$J_{s,a} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+a}{p+a}\right)^s a_k z^k , \quad (1.4)$$

$(z \in U, a \in C \setminus \underline{Z}_0,\ \underline{Z}_0 = \{0, -1, -2, \dots\}, s \in \mathbb{C}$ (see[24]).

We define the new operator

$$R^n = D^n * J_{s,a}$$

$$R^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+a}{1+a}\right)^s [1 + (k-1)\delta]^n a_k z^k \quad (1.5)$$

$$z(R^n f(z))' = \frac{1}{\delta} R^{n+1} f(z) - \frac{1-\delta}{\delta} R^n f(z). \quad (1.6)$$

Definition(1.3)[1]: Let Q the set of all functions q thatt are analytic and univalent on $\underline{U}|E(q)$, where $\underline{U} = U \cup \{z \in \partial U\}$, and

$$E(q) = \{\varepsilon \in \partial U : q(z) = \infty\} \quad (1.7)$$

and $min \left| q'(\varepsilon) \right| = p > 0$ for $\varepsilon \in \partial U \setminus E(q)$. Further, let the subclass of Q for which $q(0) = a$, be denoted yy $Q(a)$, and $Q(0) = Q_0, Q(1) = Q_1$.

The subordination methodology is applied to appropriate classes of admissible function .

The following class of admissible functions is given by Antoniuo and Miller (see[1]) .

Definition (1-4) [1]: Let Ω be a set in $\mathbb{C}$. Also $q \in Q$ and $n \in N/\{1\}, N$ being the set of positive integers. The class $H_n[\Omega, q]$ of admissible functions consists of those functions $K: \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy following admissibility conditions :

$$K(e, s, r, t) \notin \Omega,$$

whenever $e = q(\varepsilon)$, $\quad s = k\varepsilon q'(\varepsilon)$, $\quad R\left(\frac{r}{s} + 1\right) \geq kR(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)} + 1)$

and $\quad R(\frac{t}{s}) \geq k^2 R(\frac{\varepsilon^2 q'''(\varepsilon)}{q'(\varepsilon)})$ $(where\ \ z \in U, \varepsilon \in \partial U | E(q)$,$k \geq 2$).

Lemma (1.5) ,below is the foundation result in the theory of third order differential subordination .

Lemma(1.5)[1] :Let $\hat{W}[a, n]$ with $n \geq 2$ and $q \in Q_a$ satisfy the following conditons :

$$R\left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)}\right) \geq 0, \qquad \left| \frac{z p''(z)}{q'(\varepsilon)} \right| \leq k$$

$(where\ \ z \in U, \varepsilon \in \partial U | E(q)$,$k \geq 2$).If Ω is a set in $\mathbb{C}$, $\emptyset \in H_n[\Omega, q]$

and $\quad \left\{\acute{\emptyset}(p(z), zp'(z), z^2 p''(z), z^3 p^3(z); z)\right\} \subset \Omega.$

Then $\quad p(z) \prec q(z).$

The notion of the third-order differential subordination can be found in the work of Ponnurasy and Juneja [10] .The recent work by several authors (see example[4,5,6,7,8,15,22,23,26,27]) on the differential subordination attracted many researchers in this field .For example (see[2,3, 9,10,11,12,14,16,17,18,20,21,25,26,27]).

In the present paper, we investigate suitable classes of admissible functions associated with new an operator $Ŗ^n f(ẓ)$

2-Results Related to the Third-Order Subordination:

In this section , we start with a given set $Ω$,and given function q , and we determine a set of admissible new operator ,so that (1.2) holds true .For this purpose , we introduce the following new class of admissible functions which will be required to prove the main third-order differential subordination theorems for new the operator $Ŗ^n f(ẓ)$ defined by (1.5).

Definition (2.1): Let $Ω$ be a set in $\mathbb{C}$, $q \in Q_0 \cap \hat{W}_0$ .The class $Ḥ_i[Ω, q]$ of admissible functions consist of those functions:

$$\acute{\emptyset} : \mathbb{C}^4 \times U \to \mathbb{C}.$$

That satisfy the following admissibility condition: $\emptyset(ą, ɓ, ς, ȡ; z) \notin Ω$,

whenever $ą = q(ε)$, $ɓ = \delta ε q'(ε)$,

$$Re\left(\frac{ς - (1-\delta)[2ɓ - (1-\delta)ą]}{\delta^2 ɓ - \delta^2(1-\delta)ą}\right) \geq ķ Re\left(\frac{εq''(ε)}{q'(ε)} + 1\right),$$

And

$$Re\left(\frac{ȡ - 3ς - (1-\delta)[(1-\delta)ɓ + (1+\delta)ą] + 2(2-\delta)ɓ}{\delta^2 ɓ - \delta^2(1-\delta)ą}\right) \geq ķ^2 Re\left(\frac{ε^2 q'''(ε)}{q'(ε)}\right),$$

$(where\ \ ẓ \in U, ε \in \partial U | E(q)$,ķ≥ 2).

Theorem (2.1): Let $\acute{\emptyset} \in Ḥ_i[Ω, q]$.If the function $f \in A$, and $q \in Q_0$,satisfy the following conditions :

$$R\left(\frac{εq''(ε)}{q'(ε)}\right) \geq 0, \qquad \left|\frac{Ŗ^n f(ẓ)}{q'(ẓ)}\right| \leq ķ, \tag{2.1}$$

And

$$\left\{\acute{\emptyset}(Ŗ^n f(ẓ), Ŗ^{n+1} f(ẓ), Ŗ^{n+2} f(ẓ), Ŗ^{n+3} f(ẓ); ẓ): ẓ \in U\right\} \subset Ω, \tag{2.2}$$

Then

$$Ŗ^n f(ẓ) \prec q(ẓ) \quad (ẓ \in U).$$

Proof: Define the analytic funcoion $ᵽ(ẓ)$ *in U by*

$$ᵽ(ẓ) = Ŗ^n f(ẓ) . \tag{2.3}$$

From equation (1.6) and (2.3), we have

$$Ŗ^{n+1} f(ẓ) = \delta ẓ ᵽ'(ẓ) + (1-\delta) ᵽ(ẓ) . \tag{2.4}$$

By a similar argument, we get

$$Ŗ^{n+2} f(ẓ) = \delta^2 ẓ^2 ᵽ''(ẓ) + \delta(2-\delta) ẓ ᵽ'(ẓ) + (1-\delta)^2 ᵽ(ẓ), \tag{2.5}$$

and

$$Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʐ) = \boldsymbol{\delta}^{\boldsymbol{3}}ʐ^{\boldsymbol{3}}ᵽ'''(ʐ) + \boldsymbol{3\delta}^{\boldsymbol{2}}ʐ^{\boldsymbol{2}}ᵽ''(ʐ) + \boldsymbol{\delta}[(\boldsymbol{2}-\boldsymbol{\delta}) + (\boldsymbol{1}-\boldsymbol{\delta})^{\boldsymbol{2}}]ʐᵽ'(ʐ) + (\boldsymbol{1}-\boldsymbol{\delta})^{3}ᵽ(ʐ). \qquad (\boldsymbol{2.6})$$

Defin***e*** the transformati***o***n from $\mathbb{C}^{\boldsymbol{4}}$ ***to*** $\mathbb{C}$ b y

$$ą(ę, ş, ŗ, ţ) = ę\,, \qquad ɓ(ę, ş, ŗ, ţ) = \boldsymbol{\delta}ş + (\boldsymbol{1}-\boldsymbol{\delta})ę$$

$$\boldsymbol{ς}(ę, ş, ŗ, ţ) = \boldsymbol{\delta}^{\boldsymbol{2}}ŗ + \boldsymbol{\delta}(\boldsymbol{2}-\boldsymbol{\delta})ş + (\boldsymbol{1}-\boldsymbol{\delta})^{\boldsymbol{2}}ę\,, \qquad (\boldsymbol{2.7})$$

$$ȡ(ę, ş, ŗ, ţ) = \boldsymbol{\delta}^{\boldsymbol{3}}ţ + \boldsymbol{3\delta}^{\boldsymbol{2}}ŗ + \boldsymbol{\delta}[(\boldsymbol{2}-\boldsymbol{\delta}) + (\boldsymbol{1}-\boldsymbol{\delta})^{\boldsymbol{2}}]ş + (\boldsymbol{1}-\boldsymbol{\delta})^{\boldsymbol{3}}ę. \qquad (\boldsymbol{2.8})$$

Let $\qquad Ķ(ę, ş, ŗ, ţ) = Ǿ(ą, ɓ, \boldsymbol{ς}, ȡ)$

$$= Ǿ(ę, \boldsymbol{\delta}ş + (\boldsymbol{1}-\boldsymbol{\delta})ę, \boldsymbol{\delta}^{\boldsymbol{2}}ŗ + \boldsymbol{\delta}(\boldsymbol{2}-\boldsymbol{\delta})ş + (\boldsymbol{1}-\boldsymbol{\delta})^{\boldsymbol{2}}ę, \boldsymbol{\delta}^{\boldsymbol{3}}ţ + \boldsymbol{3\delta}^{\boldsymbol{2}}ŗ + \boldsymbol{\delta}[(\boldsymbol{2}-\boldsymbol{\delta}) + (\boldsymbol{1}-\boldsymbol{\delta})^{\boldsymbol{2}}]ş + (\boldsymbol{1}-\boldsymbol{\delta})^{\boldsymbol{3}}ę\,). \qquad (\boldsymbol{2.9})$$

The pr***oo***f will make us***e*** of lemma (1.5) using the ***e***quations (2.3) ,and (2.6) ,and from the ***e***quation (2.9) , ***w***e have

$$Ķ(ᵽ(ʐ), ʐ\,ᵽ'(ʐ), ʐ^{\boldsymbol{2}}ᵽ''(ʐ), ʐ^{\boldsymbol{3}}ᵽ'''(ʐ); ʐ) = Ǿ\Big(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʐ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʐ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʐ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʐ)\Big). \quad (\boldsymbol{2.10})$$

Hence ,clearly (2.2) ,becomes

$$Ķ(ᵽ(ʐ), ʐ\,ᵽ'(ʐ), ʐ^{\boldsymbol{2}}ᵽ''(ʐ), ʐ^{\boldsymbol{3}}ᵽ'''(ʐ); ʐ) \in Ω̣,$$

we note that

$$\frac{ŗ}{ş} + \boldsymbol{1} = \frac{\boldsymbol{ς} - (\boldsymbol{1}-\boldsymbol{\delta})[\boldsymbol{2}ɓ - (\boldsymbol{1}-\boldsymbol{\delta})ą]}{\boldsymbol{\delta}[ɓ - (\boldsymbol{1}-\boldsymbol{\delta})ą]},$$

and

$$\frac{ţ}{ş} = \frac{\mathrm{d} - 3\varsigma - (1-\delta)[(1-\delta)\mathrm{b} + (1+\delta)\mathrm{a}] + 2(2-\delta)\mathrm{b}}{\delta^2[\mathrm{b} - (1-\delta)\mathrm{a}]}.$$

Thus clearly ,the admissibilit*y* condition for $\acute{\emptyset} \in \mathcal{H}_i[\Omega, q]$,in definiti*o*n (2-1) is equival*e*nt to admissibilit*y* coditi*o*n $\mathrm{K} \in \mathcal{H}_2[\Omega, q]$ as given in d*e*finiti*o*n (1.4) , *w*ith $n = 2$.

Theref*o*re ,b*y* using (2.1) ,and lemma (1-5) , *w*e have

$$\mathrm{R}^n f(z) \prec q(z).$$

This proof is complete of the*o*rem (2.1).

Our ne*x*t result is consequence of the*o*rem (2.1), *w*hen the beha*v*ior of $q(z)$ *on* ∂U is n*o*t kno*w*n .

Corollary (2.1): Let $\Omega \in \mathbb{C}$,and let the functi*o*n q be unival*e*nt in U with $q(0) = 1$.*Let* $\acute{\emptyset} \in \mathcal{H}_1[\Omega, q_{\mathrm{p}}]$,f*o*r s*o*me $\mathrm{p} \in (0,1)$, *where* $q_{\mathrm{p}}(z) = q(\mathrm{p}z)$ .If the function $f \in A_{\mathrm{p}}$, ***and*** q_{p} ,satisfies the following conditions :

$$Re\left(\frac{\xi q_{\mathrm{p}}''(\xi)}{q_{\mathrm{p}}'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathrm{R}^{n+1}f(z)}{q_{\mathrm{p}}'(\xi)}\right| \leq \mathrm{k}, \qquad (z \in U, \xi \in \partial U|E(q_{\mathrm{p}}), \mathrm{k} \geq 2),$$

$$and \ \ \acute{\emptyset}\left(\mathrm{R}^n f(z), \mathrm{R}^{n+1} f(z), \mathrm{R}^{n+2} f(z), \mathrm{R}^{n+3} f(z)\right) \in \Omega.$$

Then

$$\mathrm{R}^n f(z) \prec q(z), \quad (z \in U).$$

Proof: By appl*y*ing theo*e*em (2.1) ,we get

$$\mathrm{R}^n f(z) \prec q(z) \quad (z \in U)$$

The resul*t* asserted b*y* C*o*rollary (2-1) ,is no*w* deduced from follo*w*ing sub*o*rdination propert*y*

$$q_{ꝥ}(z) \prec q(z), \quad (z \in U).$$

This pr***oo***f is complet***e*** of corollar***y*** (2.1).

If $\Omega \neq \mathbb{C}$,is simpl***y*** –connect***e***d d***o***main ,th***e*** $\Omega = M(U)$,for s***o***me c***o***nformal mapping $M(z)$ ***of U on to*** Ω .In this cas***e*** the class $H_i[M(U), q]$ is ***w***ritten as $H_i[M, q]$.

This lead***s*** to the follo***w***ing immediat***e*** cons***e***quence of the***o***rem (2.1).

<u>Theorem(2.2):</u> L***e***t $\phi \in H_i[M, q]$.If the functi***o***n $f \in A$,and $q \in Q_0$, ***s***atisfy the follo***w***ing c***o***nditi***o***ns :

$$Re\left(\frac{\varepsilon q_{ꝥ}''(\varepsilon)}{q_{ꝥ}'(\varepsilon)}\right) \geq 0, \qquad \left|\ \frac{R^{n+1}f(z)}{q_{ꝥ}'(z)}\ \right| \leq k, \qquad (2.11)$$

and

$$\phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z) \prec M(z), \qquad (2.12)$$

then

$$R^n f(z) \prec q(z), \quad (z \in U).$$

Th***e*** ne***x***t re***s***ult is an imm***e***diate c***o***nsequence of corollar***y*** (2.1) .

<u>Corollary (2.2):</u> L***e***t $\Omega \in \mathbb{C}$,and l***e***t th***e*** functi***o***n q be univalen***t*** in U ***w***ith $q(0) = 1$ .Let $\phi \in H_1[\Omega, q_{ꝥ}]$,f***o***r ***s***ome $ꝥ \in (0,1)$, ***where*** $q_{ꝥ}(z) = q(ꝥz)$ .If th***e*** functi***o***n $f \in A$, ***and*** $q_{ꝥ}$, ***s***atisfies the follo***w***ing c***o***nditions :

$$Re\left(\frac{\varepsilon q_{ꝥ}''(\varepsilon)}{q_{ꝥ}'(\varepsilon)}\right) \geq 0, \qquad \left|\ \frac{R^{n+1}f(z)}{q_{ꝥ}'(\varepsilon)}\ \right| \leq k \qquad (z \in U, \varepsilon \in \partial U | E(q_{ꝥ}), k \geq 2),$$

and $\phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z) \prec M(z),$

then

$$Ŗ^{n}\boldsymbol{f}(ẕ) \prec \boldsymbol{q}(ẕ) \quad (ẕ \in \boldsymbol{U}).$$

The following result yield the best domianant of differential subordination (2.12) .

Theorem(2.3): Let the function Ṃ be univalent in U .Also $\acute{Ø}: \mathbb{C}^{4} \times \boldsymbol{U} \to \mathbb{C}$,and Ķ given by (2.9) .Suppose that following differential equation

$$Ķ(ᵽ(ẕ), ẕ\, ᵽ'(ẕ), ẕ^{2}ᵽ''(ẕ), ẕ^{3}ᵽ'''(ẕ); ẕ) = Ṃ(ẕ), \quad (\mathbf{2.13})$$

has a solution q(ẕ) with q(0)=1 , which satisfies the condition (2.1) .If $\boldsymbol{f} \in \boldsymbol{A}$, satisfies the condition (2.12) ,and if

$$\acute{Ø}(\, Ŗ^{n}\boldsymbol{f}(ẕ), Ŗ^{n+1}\boldsymbol{f}(ẕ), Ŗ^{n+2}\boldsymbol{f}(ẕ), Ŗ^{n+3}\boldsymbol{f}(ẕ); ẕ) \;\boldsymbol{is\ analytic\ in\ U},$$

then

$$Ŗ^{n}\boldsymbol{f}(ẕ) \prec \boldsymbol{q}(ẕ), \quad (ẕ \in \boldsymbol{U})$$

and *q(ẕ)* is the best dominant .

Proof: From theorem (2.1) , we see that q is a dominant of (2.12) .Since q satisfies a solution of (2.13).

Therefore ,q will be dominant by all dominants .Hence q ,is the best dominant .This completes the proof of theorem (2.3) .

In view of definition (2.1) ,and in special case when $\boldsymbol{q}(ẕ) = Ģẕ\ (Ģ > \mathbf{0})$, $\boldsymbol{the\ class}\ Ḥ_{\boldsymbol{i}}[Ώ, \boldsymbol{q}]$,of admissible functions ,denoted by $Ḥ_{\boldsymbol{i}}[Ώ, Ģ]$ is expressed follows .

Definition(2.2); Let *Ώ* be set in $\mathbb{C}$,and $Ģ > \mathbf{0}$ .The class $Ḥ_{i}[Ώ, Ģ]$,of admissible functions consists of those functions $\acute{Ø}: \mathbb{C}^{4} \times \boldsymbol{U} \to \mathbb{C}$ such that

$$\begin{aligned}\acute{Ø}(Ģ\boldsymbol{e}^{i\theta}, [\boldsymbol{\delta}Ḍ + (\mathbf{1}-\boldsymbol{\delta})]Ģ\boldsymbol{e}^{i\theta}, \boldsymbol{\delta}^{2}Ẹ + [\boldsymbol{\delta}(\mathbf{2}-\boldsymbol{\delta})Ḍ + (\mathbf{1}-\boldsymbol{\delta})^{2}]Ģ\boldsymbol{e}^{i\theta}, \boldsymbol{\delta}^{3}Ḻ + \mathbf{3}\boldsymbol{\delta}^{2}Ẹ \\ + (\boldsymbol{\delta}[(\mathbf{2}-\boldsymbol{\delta}) + (\mathbf{1}-\boldsymbol{\delta})^{2}]Ḍ + (\mathbf{1}-\boldsymbol{\delta})^{3})Ģ\boldsymbol{e}^{i\theta}\,) \\ \notin Ώ\,, \qquad (\mathbf{2.14})\end{aligned}$$

where $(z \in \boldsymbol{U})$,

$$\boldsymbol{Re}(E\boldsymbol{e}^{-i\theta}) \geq (k-\mathbf{1})kG$$

and $\boldsymbol{Re}(L\boldsymbol{e}^{-i\theta}) \geq \mathbf{0}$, $\forall\boldsymbol{\theta} \in \boldsymbol{R}$, $k \geq \mathbf{2}$.

Corollary(2.3); Let $\acute{\emptyset} \in H_i[\Omega, G]$ .If the functi***o***n $\boldsymbol{f} \in \boldsymbol{A}$, ***s***atisfies the follo***w***ing c***o***nditions

$$|R^{\boldsymbol{n}}\boldsymbol{f}(z)| \leq kG, \qquad (z \in \boldsymbol{U}; k \geq \mathbf{2}; G > \mathbf{0}),$$

and

$$(R^{\boldsymbol{n}}\boldsymbol{f}(z), R^{\boldsymbol{n}+\mathbf{1}}\boldsymbol{f}(z), R^{\boldsymbol{n}+\mathbf{2}}\boldsymbol{f}(z), R^{\boldsymbol{n}+\mathbf{3}}\boldsymbol{f}(z); z) \in \Omega,$$

then $|R^{\boldsymbol{n}}\boldsymbol{f}(z)| < G,$

In special case, ***w***hen $\Omega = \boldsymbol{q}(\boldsymbol{U}) = \{\omega: |\omega| < G\}$, ***the class*** $H_i[\Omega, G]$ is ***s***imple denoted b***y*** $H_i[G]$ .Corollar***y*** (2.3) can b***e*** re***w***ritten in th***e*** follo***w***ing from .

Corollary(2.4); L***e***t $\acute{\emptyset} \in H_i[G]$ .If th***e*** functi***o***n $\boldsymbol{f} \in \boldsymbol{A}$, ***s***atisfies th***e*** follo***w***ing conditi***o***ns:

$$|R^{\boldsymbol{n}}\boldsymbol{f}(z)| \leq kG, \qquad (z \in \boldsymbol{U}; k \geq \mathbf{2}; G > \mathbf{0}),$$

and

$$|R^{\boldsymbol{n}}\boldsymbol{f}(z), R^{\boldsymbol{n}+\mathbf{1}}\boldsymbol{f}(z), R^{\boldsymbol{n}+\mathbf{2}}\boldsymbol{f}(z), R^{\boldsymbol{n}+\mathbf{3}}\boldsymbol{f}(z); z| < G,$$

then $|R^{\boldsymbol{n}}\boldsymbol{f}(z)| < G.$

Corollary (2.5); L***e***t $k \geq \mathbf{2}, \mathbf{0} \neq \boldsymbol{q} \in \boldsymbol{C}$, ***and*** $G > \mathbf{0}$. If th***e*** functi***o***n $\boldsymbol{f} \in \boldsymbol{A}$,satisfie***s*** the follo***w***ing c***o***nditions:

$$|R^{\boldsymbol{n}+\mathbf{1}}\boldsymbol{f}(z)| \leq kG$$

and

$$|Ŗ^{n+1}f(ʐ) - Ŗ^{n}f(ʐ)| < Ģʐ,$$

then $|Ŗ^{n}f(ʐ)| < Ģ.$

Proof; Let $\acute{Ø}(ą, ɓ, \boldsymbol{ς}, ȡ) = ɓ - ą$, $\boldsymbol{and}$ $Ϙ = Ṃ(\boldsymbol{U}), \boldsymbol{where}$

$$Ṃ(ʐ) = Ģʐ \ \ (Ģ > \mathbf{0}).$$

Us***e*** corollar***y*** (2.3) , ***w***e need to sho***w*** that $\acute{Ø} \in Ḫ_{\boldsymbol{i}}[Ϙ, Ģ]$,that is the admissibilit***y*** condition (2.14) is ***s***atisfied .This follo***w***s readil***y*** , ***s***ince it is ***s***een that

$$\left|\acute{Ø}(ą, ɓ, \boldsymbol{ς}, ȡ; ʐ)\right| = |ķ - \mathbf{1}| \geq Ģ,$$

where $ʐ \in \boldsymbol{U}$,$\forall \boldsymbol{\theta} \in \boldsymbol{R}$, $\boldsymbol{and}$ $ķ \geq \mathbf{0}$ .The r***e***quired result no***w*** follo***w***s from corollar***y*** (2.3) .This complete***s*** the pr***oo***f .

<u>**Definition(2.3)**:</u> L***e***t Ϙ b***e*** a s***e***t in ℂ, $\boldsymbol{q} \in \boldsymbol{Q_1} \cap Ḁ_{\mathbf{1}}$ .The class $Ḫ_{\boldsymbol{i},\mathbf{1}}[Ϙ, \boldsymbol{q}]$ of admissibl***e*** functi***o***ns consist***s*** of th***o***se functi***o***n $\acute{Ø} : ℂ^{\mathbf{4}} \times \boldsymbol{U} \rightarrow ℂ$,which satisf***y*** the follo***w***ing admissibilit***y*** conditi***o***ns:

$$Ø(ą, ɓ, \boldsymbol{ς}, ȡ; \boldsymbol{z}) \notin Ϙ,$$

whenever $ą = \boldsymbol{q}(ɛ)$, $ɓ = \boldsymbol{\delta}ɛ\boldsymbol{q}'(ɛ) + \boldsymbol{q}(ɛ),$

$$\boldsymbol{Re}\left(\frac{\boldsymbol{ς} - \mathbf{2}ɓ - (\mathbf{1} - \mathbf{2}\boldsymbol{\delta})ą}{\boldsymbol{\delta}(ɓ - ą)}\right) \geq ķ\boldsymbol{R}(\frac{ɛ\boldsymbol{q}''(ɛ)}{\boldsymbol{q}'(ɛ)} + \mathbf{1})$$

and

$$\boldsymbol{Re}\left(\frac{ȡ - (\mathbf{3} + \mathbf{3}\boldsymbol{\delta})\boldsymbol{ς} - \big([\boldsymbol{\delta}(\mathbf{3} + \boldsymbol{\delta}) + \mathbf{3}] + (\mathbf{3} + \mathbf{3}\boldsymbol{\delta})(\mathbf{2} - \boldsymbol{\delta})\big)ɓ}{\boldsymbol{\delta}^{\mathbf{2}}(ɓ - ą)} - \frac{[(\mathbf{3} + \mathbf{3}\boldsymbol{\delta})(\mathbf{1} - \boldsymbol{\delta}) - \boldsymbol{\delta}(\mathbf{3} + \boldsymbol{\delta}) - \mathbf{2}]ą}{\boldsymbol{\delta}^{\mathbf{2}}(ɓ - ą)}\right) \geq ķ^{\mathbf{2}}\boldsymbol{Re}\left(\frac{ɛ^{\mathbf{2}}\boldsymbol{q}'''(ɛ)}{\boldsymbol{q}'(ɛ)}\right),$$

$(\boldsymbol{where}\ \ ẓ \in \boldsymbol{U}, ẹ \in \boldsymbol{\partial U}|\boldsymbol{E}(\boldsymbol{q})$,ķ$\geq \mathbf{2}$).

Theorem(2.4): Let $Ǿ \in Ḫ_{i,1}[Ω̧, \boldsymbol{q}]$.If the function $\boldsymbol{f} \in \boldsymbol{A}$, *and* $\boldsymbol{q} \in \boldsymbol{Q_1}$,satisfythe following conditions :

$$\boldsymbol{Re}\left(\frac{ẹ\boldsymbol{q}''(ẹ)}{\boldsymbol{q}'(ẹ)}\right) \geq \mathbf{0}, \qquad \left| \frac{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ẓ)}{ẓ\boldsymbol{q}'(ẓ)} \right| \leq ķ \qquad (\mathbf{2.15})$$

and

$$\left\{Ǿ\left(\frac{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ẓ)}{ẓ}, \frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ẓ)}{ẓ}, \frac{Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ẓ)}{ẓ}, \frac{Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ẓ)}{ẓ}; ẓ\right) : ẓ \in Ω̧\right\} \subset Ω̧, \qquad (\mathbf{2.16})$$

then

$$\frac{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ẓ)}{ẓ} \prec \boldsymbol{q}(ẓ), \quad (ẓ \in \boldsymbol{U}).$$

Proof: Define the analytic function $ᵽ(ẓ)$ $\boldsymbol{in\ U}$ by

$$ᵽ(ẓ) = \frac{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ẓ)}{ẓ}. \qquad (\mathbf{2.17})$$

From equation (1.6) and (2.3), we have

$$\frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ẓ)}{ẓ} = \boldsymbol{\delta}ẓᵽ'(ẓ) + ᵽ(ẓ). \qquad (\mathbf{2.18})$$

By a similar argument, we get

$$\frac{Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ẓ)}{ẓ} = \boldsymbol{\delta}^2ẓ^2ᵽ''(ẓ) + \boldsymbol{\delta}(\mathbf{2} + \boldsymbol{\delta})ẓᵽ'(ẓ) + ᵽ(ẓ), \qquad (\mathbf{2.19})$$

and

$$\frac{Ŗ^{n+3}\boldsymbol{f}(ƺ)}{ƺ} = \boldsymbol{\delta}^3 ƺ^3 ᵽ'''(ƺ) + \boldsymbol{\delta}^2(\mathbf{3}+\mathbf{3}\boldsymbol{\delta})ƺ^2 ᵽ''(ƺ) + \boldsymbol{\delta}[(\mathbf{3}+\boldsymbol{\delta})+\mathbf{3}]ƺᵽ'(ƺ) + ᵽ(ƺ). \qquad (\mathbf{2.20})$$

Defin***e*** the transf***o***rmation fr***o***m $\mathbb{C}^{\mathbf{4}}$ ***to*** $\mathbb{C}$ b***y***

$$ą(ȩ,ş,ŗ,ţ) = ȩ\,, \qquad ɓ(ȩ,ş,ŗ,ţ) = \boldsymbol{\delta}ş + ȩ$$

$$\boldsymbol{\varsigma}(ȩ,ş,ŗ,ţ) = \boldsymbol{\delta}^{\mathbf{2}}ŗ + \boldsymbol{\delta}(\mathbf{2}+\boldsymbol{\delta})ş + ȩ\,, \qquad (\mathbf{2.21})$$

and

$$ȡ(ȩ,ş,ŗ,ţ) = \boldsymbol{\delta}^{\mathbf{3}}ţ + \boldsymbol{\delta}^{\mathbf{2}}(\mathbf{3}+\mathbf{3}\boldsymbol{\delta})ŗ + \boldsymbol{\delta}[\boldsymbol{\delta}(\mathbf{3}+\boldsymbol{\delta})+\mathbf{3}]ş + ȩ. \; (\mathbf{2.22})$$

Let $Ķ(ȩ,ş,ŗ,ţ) = Ǿ(ą,ɓ,\boldsymbol{\varsigma},ȡ)$

$$= Ǿ(ȩ, \boldsymbol{\delta}ş + ȩ, \boldsymbol{\delta}^{\mathbf{2}}ŗ + \boldsymbol{\delta}(\mathbf{2}+\boldsymbol{\delta})ş + ȩ, \boldsymbol{\delta}^{\mathbf{3}}ţ + \boldsymbol{\delta}^{\mathbf{2}}(\mathbf{3}+\mathbf{3}\boldsymbol{\delta})ŗ + \boldsymbol{\delta}[\boldsymbol{\delta}(\mathbf{3}+\boldsymbol{\delta})+\mathbf{3}]ş + ȩ). \qquad (\mathbf{2.23})$$

The pr***oo***f will make us***e*** of lemma (1.5) using the ***e***quations (2.17) ,and (2.19) ,and from the ***e***quation (2.23) , ***w***e have

$$Ķ(ᵽ(ƺ), ƺ\,ᵽ'(ƺ), ƺ^{\mathbf{2}}ᵽ''(ƺ), ƺ^{\mathbf{3}}ᵽ'''(ƺ); ƺ) = Ǿ\left(\frac{Ŗ^{n}\boldsymbol{f}(ƺ)}{ƺ}, \frac{Ŗ^{n+1}\boldsymbol{f}(ƺ)}{ƺ}, \frac{Ŗ^{n+2}\boldsymbol{f}(ƺ)}{ƺ}, \frac{Ŗ^{n+3}\boldsymbol{f}(ƺ)}{ƺ}; ƺ\right). \qquad (\mathbf{2.24})$$

Hence ,clearly (2.16) ,becomes

$$Ķ(ᵽ(ƺ), ƺ\,ᵽ'(ƺ), ƺ^{\mathbf{2}}ᵽ''(ƺ), ƺ^{\mathbf{3}}ᵽ'''(ƺ); ƺ) \in Ȣ,$$

we note that

$$\frac{ŗ}{ş} + \mathbf{1} = \frac{\boldsymbol{\varsigma} - \mathbf{2}ɓ + (\mathbf{1} - \mathbf{2}\boldsymbol{\delta})ą}{\boldsymbol{\delta}(ɓ - ą)},$$

and

$$\frac{t}{s} = \frac{d - (3+3\delta)\varsigma - \big([\delta(3+\delta)+3] + (3+3\delta)(2-\delta)\big)b}{\delta^2(b-a)} - \frac{[(3+3\delta)(1-\delta) - \delta(3+\delta) - 2]a}{\delta^2(b-a)}.$$

Thus clearly ,the admissibility condition for $\phi \in H_{i,1}[\Omega, q]$,in definition (2-3) is equivalent to admissibility codition $K \in H_2[\Omega, q]$ as given in definition (1.4) , with $n = 2$.Therefore ,by using (2.15) ,and lemma (1.5), we have

$$\frac{R^n f(z)}{z} \prec q(z).$$

If $\Omega \neq \mathbb{C}$,is simply connected domain ,the $\Omega = M(U)$,for some conformal mapping $M(z)$ ***of U on to*** Ω .In this case the class $H_{i,1}[M(U), q]$ is written as $H_{i,1}[M, q]$.

This leads to the following immediate consequence of theorem (2.4) ,is stated below .

<u>Theorem(2.5):</u> Let $\phi \in H_{i,1}[M, q]$.If the function $f \in A$,and $q \in Q_1$,satisfy the following conditions :

$$R\left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)}\right) \geq 0, \qquad \left|\frac{R^{n+1} f(z)}{z q'(\varepsilon)}\right| \leq k, \tag{2.25}$$

and

$$\phi\left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z\right) \prec M(z), \tag{2.26}$$

then

$$\frac{R^n f(z)}{z} \prec q(z) \quad (z \in U).$$

In view of definition (2.3) ,and in special case when $q(z) = Gz$ $(G > 0)$, ***the class*** $H_i[\Omega, q]$,of admissible functions ,denoted by $H_{i,1}[\Omega, G]$ is expressed follows .

<u>Definition(2.4);</u> Let Ω be **s**et in $\mathbb{C}$,and $G > \mathbf{0}$ .The class $H_{i,1}[\Omega, G]$,of admissibl**e** functions c**o**nsists of th**o**se functions $\phi: \mathbb{C}^4 \times \boldsymbol{U} \to \mathbb{C}$ such that:

$$\phi(G\boldsymbol{e^{i\theta}}, [\boldsymbol{\delta}D + \mathbf{1}]G\boldsymbol{e^{i\theta}}, \boldsymbol{\delta^2}E + [\boldsymbol{\delta}(\mathbf{2} + \boldsymbol{\delta})D + \mathbf{1}]G\boldsymbol{e^{i\theta}}, \boldsymbol{\delta^3}L + \boldsymbol{\delta^2}(\mathbf{3} + \mathbf{3}\boldsymbol{\delta})E + (\boldsymbol{\delta}[\boldsymbol{\delta}(\mathbf{3} + \boldsymbol{\delta}) + \mathbf{3}]D + \mathbf{1})G\boldsymbol{e^{i\theta}}) \notin \Omega, \quad (\mathbf{2.27})$$

where $(z \in \boldsymbol{U}), \boldsymbol{Re}(E\boldsymbol{e^{-i\theta}}) \geq (k - \mathbf{1})kG$

and $\quad \boldsymbol{Re}(L\boldsymbol{e^{-i\theta}}) \geq \mathbf{0}, \quad \forall \boldsymbol{\theta} \in \boldsymbol{R}, k \geq \mathbf{2}$.

<u>Corollary(2.6);</u> Let $\phi \in H_{i,1}[\Omega, G]$ .If the functio**n** $\boldsymbol{f} \in \boldsymbol{A}$, satisfie**s** the follo**w**ing c**o**nditions:

$$\left|\frac{R^{n+1}\boldsymbol{f}(z)}{z}\right| \leq kG, \quad (z \in \boldsymbol{U}; k \geq \mathbf{2}; G > \mathbf{0}),$$

and

$$\phi\left(\frac{R^{n}\boldsymbol{f}(z)}{z}, \frac{R^{n+1}\boldsymbol{f}(z)}{z}, \frac{R^{n+2}\boldsymbol{f}(z)}{z}, \frac{R^{n+3}\boldsymbol{f}(z)}{z}; z\right) \in \Omega,$$

then $\quad \left|\frac{R^{n}f(z)}{z}\right| < G.$

In **s**pecial case, **w**hen $\Omega = \boldsymbol{q}(\boldsymbol{U}) = \{\omega: |\omega| < G\}$, ***the class*** $H_{i,1}[\Omega, G]$ is simpl**y** denot**e**d by $H_{i,1}[G]$ .Corollar**y** (2.6) can b**e** re**w**ritten in the follo**w**ing form .

<u>Corollary (2.7);</u> Let $\phi \in H_{i,1}[\Omega, G]$ .If the functi**o**n $\boldsymbol{f} \in \boldsymbol{A}$,satisfi**e**s the follo**w**ing condition**s**:

$$\left|\frac{R^{n}\boldsymbol{f}(z)}{z}\right| \leq kG, \quad (z \in \boldsymbol{U}; k \geq \mathbf{2}; G > \mathbf{0}),$$

and

$$\left|\phi\left(\frac{R^{n}\boldsymbol{f}(z)}{z}, \frac{R^{n+1}\boldsymbol{f}(z)}{z}, \frac{R^{n+2}\boldsymbol{f}(z)}{z}, \frac{R^{n+3}\boldsymbol{f}(z)}{z}; z\right)\right| < G,$$

then $\left|\frac{R^{n}f(z)}{z}\right| < G.$

Definition(2.5): Let Ω be a ***s***et in $\mathbb{C}$, $\boldsymbol{q} \in \boldsymbol{Q_1} \cap A_1$ .The class $H_{i,2}[\Omega, \boldsymbol{q}]$ of admissibl***e*** function***s*** consist***s*** of th***o***se functi***o***ns $\phi : \mathbb{C}^4 \times \boldsymbol{U} \to \mathbb{C}$,which satisf***y*** the follo***w***ing admissibili***y*** c***o***nditions: $\phi(a, b, \boldsymbol{\varsigma}, d; \boldsymbol{z}) \notin \Omega$,

whenever $a = \boldsymbol{q}(\varepsilon)$, $b = \frac{\delta\varepsilon\boldsymbol{q}'(\varepsilon)+(\boldsymbol{q}(\varepsilon))^2}{\boldsymbol{q}(\varepsilon)}$,

$$\boldsymbol{Re}\left(\frac{\boldsymbol{\varsigma}b + \boldsymbol{2}a^{\boldsymbol{2}} - \boldsymbol{3}ab}{\boldsymbol{\delta}(b - a)}\right) \geq k\boldsymbol{R}\left(\frac{\varepsilon\boldsymbol{q}''(\varepsilon)}{\boldsymbol{q}'(\varepsilon)} + \boldsymbol{1}\right),$$

and

$$\boldsymbol{Re}([(b\boldsymbol{\varsigma} - ab + a^{\boldsymbol{2}} + \boldsymbol{\delta}a][bd - b\boldsymbol{\varsigma} + ab - a^{\boldsymbol{2}} - \boldsymbol{\delta}a] - b^{\boldsymbol{3}}(\boldsymbol{\varsigma} - b) - [b(\boldsymbol{\varsigma} - \boldsymbol{\delta} - \boldsymbol{3}a) + a(\boldsymbol{\delta} + \boldsymbol{2}a)][b(\boldsymbol{5}b - \boldsymbol{5}a - \boldsymbol{\varsigma} + \boldsymbol{2\delta}) + a(\boldsymbol{\delta} + \boldsymbol{2}a) + \frac{a^{\boldsymbol{2}}}{\boldsymbol{\delta}^{\boldsymbol{3}}}[(b(\boldsymbol{\varsigma} - \boldsymbol{3}a - \boldsymbol{\delta}) + a(\boldsymbol{\delta} + \boldsymbol{2}a)]^{\boldsymbol{2}} - (b - a)[ab(\boldsymbol{\delta}^{\boldsymbol{2}} - a + \boldsymbol{4}b) + b(\boldsymbol{\delta}^{\boldsymbol{2}} + \boldsymbol{\delta}b - \boldsymbol{2}b^{\boldsymbol{2}}) - a(\boldsymbol{\delta} + \boldsymbol{\delta}a + \boldsymbol{\delta}^{\boldsymbol{2}}a - a^{\boldsymbol{2}})]] * [\boldsymbol{\delta}^{\boldsymbol{2}}(b - a)]^{-\boldsymbol{1}}) \geq k^{\boldsymbol{2}}\boldsymbol{Re}(\frac{\varepsilon^{\boldsymbol{2}}\boldsymbol{q}'''(\varepsilon)}{\boldsymbol{q}'(\varepsilon)}),$$

$(\boldsymbol{where}\ \ z \in \boldsymbol{U}, \varepsilon \in \boldsymbol{\partial U} \backslash \boldsymbol{E}(\boldsymbol{q}), k \geq \boldsymbol{2}\ )$.

Theorem(2.6): L**e**t $\phi \in H_{i,2}[\Omega, \boldsymbol{q}]$.If th**e** functi**o**n $\boldsymbol{f} \in \boldsymbol{A}$, and $\boldsymbol{q} \in \boldsymbol{Q_1} \cap A_1$,satisf**y** the follo**w**ing c**o**nditions *:*

$$\boldsymbol{Re}\left(\frac{\varepsilon\boldsymbol{q}''(\varepsilon)}{\boldsymbol{q}'(\varepsilon)}\right) \geq \boldsymbol{0}, \qquad \left|\ \frac{R^{\boldsymbol{n+2}}\boldsymbol{f}(z)}{R^{\boldsymbol{n+1}}\boldsymbol{f}(z)}\ \right| \leq k, \qquad (\boldsymbol{2.28})$$

and

$$\left\{\phi\left(\frac{R^{\boldsymbol{n+1}}\boldsymbol{f}(z)}{R^{\boldsymbol{n}}\boldsymbol{f}(z)}, \frac{R^{\boldsymbol{n+2}}\boldsymbol{f}(z)}{R^{\boldsymbol{n+1}}\boldsymbol{f}(z)}, \frac{R^{\boldsymbol{n+3}}\boldsymbol{f}(z)}{R^{\boldsymbol{n+2}}\boldsymbol{f}(z)}, \frac{R^{\boldsymbol{n+4}}\boldsymbol{f}(z)}{R^{\boldsymbol{n+3}}\boldsymbol{f}(z)}; z\right) : z \in \boldsymbol{U}\right\} \subset \Omega\,, \qquad (\boldsymbol{2.29})$$

then

$$\frac{Ŗ^{n+1}f(z)}{Ŗ^{n}f(z)} \prec q(z), \quad (z \in U).$$

Proof: Define the anayytic function $ᵽ(z)$ $in\ U$ by

$$ᵽ(z)=\frac{Ŗ^{n+1}f(z)}{Ŗ^{n}f(z)}. \qquad (2.30)$$

From equation (1.6) and (2.30), we have

$$\frac{Ŗ^{n+2}f(z)}{Ŗ^{n+1}f(z)}=\frac{\delta zᵽ'(z)+(ᵽ(z))^2}{ᵽ(z)}=V. \qquad (2.31)$$

By a similar argument, we get

$$\frac{Ŗ^{n+3}f(z)}{Ŗ^{n+2}f(z)}=Ḅ, \qquad (2.32)$$

and

$$\frac{Ŗ^{n+4}f(z)}{Ŗ^{n+3}f(z)}=Ḅ+Ḅ^{-1}[Ḙ+V^{-1}Ḍ-V^{-2}Ḙ^{2}], (2.33)$$

where

$$Ḅ=\frac{\delta zᵽ'(z)+(ᵽ(z))^2}{ᵽ(z)}+\frac{\frac{\delta^2 z^2 ᵽ''(z)+\delta^2 zᵽ'(z)}{ᵽ(z)}-\left(\frac{\delta zᵽ'(z)}{ᵽ(z)}\right)^2+\delta zᵽ(z)}{\frac{\delta zᵽ'(z)+(ᵽ(z))^2}{ᵽ(z)}},$$

$$Ḙ=\frac{\delta^2 z^2 ᵽ''(z)+\delta^2 zᵽ'(z)}{ᵽ(z)}-\left(\frac{\delta zᵽ'(z)}{ᵽ(z)}\right)^2+\delta zᵽ'(z),$$

and

$$Ḍ = \frac{\boldsymbol{\delta}^{3}ȥ^{3}ᵽ'''(ȥ) + \mathbf{3}\boldsymbol{\delta}^{2}ȥ^{2}ᵽ''(ȥ) + \boldsymbol{\delta}^{3}ȥᵽ'(ȥ)}{ᵽ(ȥ)} - \frac{\mathbf{3}\boldsymbol{\delta}^{3}ȥ^{3}ᵽ'(ȥ)ᵽ''(ȥ) + \mathbf{3}\big(\boldsymbol{\delta}ȥᵽ'(ȥ)\big)^{2}}{\big(ᵽ(ȥ)\big)^{2}}$$
$$+ \mathbf{2}\left(\frac{\boldsymbol{\delta}ȥᵽ'(ȥ)}{ᵽ(ȥ)}\right)^{2} + \boldsymbol{\delta}^{2}ȥ^{2}ᵽ''(ȥ) + \boldsymbol{\delta}^{2}ȥᵽ'(ȥ).$$

We no**w** define the transf**o**rmation from $\mathbb{C}^{4}$ ***to*** $\mathbb{C}$ *b y*

$$ą(ę, ș, ŗ, ţ) = ę \qquad , ɓ(ę, ș, ŗ, ţ) = \frac{\boldsymbol{\delta}ș + ę^{2}}{ę} = Ș,$$

$$\boldsymbol{\varsigma}(ę, ș, ŗ, ţ) = \frac{\boldsymbol{\delta}ș + ę^{2}}{ę} + \frac{\frac{\boldsymbol{\delta}^{2}ŗ + \boldsymbol{\delta}^{2}ș}{ę} - \left(\frac{\boldsymbol{\delta}ș}{ę}\right)^{2} + \boldsymbol{\delta}ș}{\frac{\boldsymbol{\delta}ș + ę^{2}}{ę}} = Ç\,, \qquad (\mathbf{2.34})$$

and

$$ȡ(ę, ș, ŗ, ţ) = Ç + Ç^{-\mathbf{1}}[Ḷ + Ģ^{-\mathbf{1}}Ș - Ș^{-\mathbf{2}}Ḷ^{\mathbf{2}}], \qquad (\mathbf{2.35})$$

where $Ḷ = \frac{\boldsymbol{\delta}^{2}ŗ + \boldsymbol{\delta}^{2}ș}{ę} - \left(\frac{\boldsymbol{\delta}ș}{ę}\right)^{\mathbf{2}} + \boldsymbol{\delta}ș$

and

$$Ģ = \frac{\boldsymbol{\delta}^{3}ţ + \mathbf{3}\boldsymbol{\delta}^{3}ŗ + \boldsymbol{\delta}^{3}ș}{ę} - \frac{\mathbf{3}\boldsymbol{\delta}^{3}ŗș + \mathbf{3}\boldsymbol{\delta}^{3}ș^{2}}{ę} + \mathbf{2}\left(\frac{\boldsymbol{\delta}ș}{ę}\right)^{3} + \boldsymbol{\delta}^{2}ŗ + \boldsymbol{\delta}^{2}ș$$

Let $Ķ(ę, ș, ŗ, ţ) = Ǿ(ą, ɓ, \boldsymbol{\varsigma}, ȡ)$

$$= Ǿ(ę, Ș, Ç, Ç + Ç^{-\mathbf{1}}[Ḷ + Ģ^{-\mathbf{1}}Ș - Ș^{-\mathbf{2}}Ḷ^{\mathbf{2}}]). \qquad (\mathbf{2.36})$$

The pr***oo***f ***w***ill mak***e*** us***e*** of lemma (1.5) ***u***sing the ***e***quations (2.30), and (2.33) ,and fr***o***m the equation (2.36) , ***w***e have

$$Ķ(ᵽ(ʐ), ʐ\,ᵽ'(ʐ), ʐ^2ᵽ''(ʐ), ʐ^3ᵽ'''(ʐ); ʐ) = \acute{Ø}\left(\frac{Ŗ^{n+1}f(ʐ)}{Ŗ^{n}f(ʐ)}, \frac{Ŗ^{n+2}f(ʐ)}{Ŗ^{n+1}f(ʐ)}, \frac{Ŗ^{n+3}f(ʐ)}{Ŗ^{n+2}f(ʐ)}, \frac{Ŗ^{n+4}f(ʐ)}{Ŗ^{n+3}f(ʐ)}; ʐ\right). \quad (\mathbf{2.37})$$

Hence ,clearly (2.29) ,becomes

$$Ķ(ᵽ(ʐ), ʐ\,ᵽ'(ʐ), ʐ^2ᵽ''(ʐ), ʐ^3ᵽ'''(ʐ); ʐ) \in Ω̦.$$

We note that

$$\frac{ŗ}{ş} + \mathbf{1} = \frac{\boldsymbol{ς}ɓ + \mathbf{2}ą^{\mathbf{2}} - \mathbf{3}ɓą}{\boldsymbol{\delta}(ɓ - ą)},$$

and

$$\frac{ţ}{ş} = [(ɓ\boldsymbol{ς} - ąɓ + ą^{\mathbf{2}} + \boldsymbol{\delta}ą][ɓȡ - ɓ\boldsymbol{ς} + ąɓ - ą^{\mathbf{2}} - \boldsymbol{\delta}ą] - ɓ^{\mathbf{3}}(\boldsymbol{ς} - ɓ) - [ɓ(\boldsymbol{ς} - \boldsymbol{\delta} - \mathbf{3}ą) + ą(\boldsymbol{\delta} + \mathbf{2}ą)][ɓ(\mathbf{5}ɓ - \mathbf{5}ą - \boldsymbol{ς} + \mathbf{2}\boldsymbol{\delta}) + ą(\boldsymbol{\delta} + \mathbf{2}ą) + \frac{ą^{\mathbf{2}}}{\boldsymbol{\delta}^{\mathbf{3}}}[(ɓ(\boldsymbol{ς} - \mathbf{3}ą - \boldsymbol{\delta}) + ą(\boldsymbol{\delta} + \mathbf{2}ą)]^{\mathbf{2}} - (ɓ - ą)[ąɓ(\boldsymbol{\delta}^{\mathbf{2}} - ą + \mathbf{4}ɓ) + ɓ(\boldsymbol{\delta}^{\mathbf{2}} + \boldsymbol{\delta}ɓ - \mathbf{2}ɓ^{\mathbf{2}}) - ą(\boldsymbol{\delta} + \boldsymbol{\delta}ą + \boldsymbol{\delta}^{\mathbf{2}}ą - ą^{\mathbf{2}})]] * [\boldsymbol{\delta}^{\mathbf{2}}(ɓ - ą)]^{-\mathbf{1}}.$$

Thus clear**y** ,the admissibilit**y** conditi**o**n for $\acute{Ø} \in Ḩ_{i,2}[Ω̦, \boldsymbol{q}]$,in definiti**o**n (2.5) is equivalent**t**to admissibilit**y** condition $Ķ \in Ḩ_{\mathbf{2}}[Ω̦, \boldsymbol{q}]$ as given in definition (1.4) , **w**ith $\boldsymbol{n} = \mathbf{2}$.Therefore ,b**y** using (2.30) ,and lemma (1.5) , **w**e have

$$\frac{Ŗ^{n+1}f(ʐ)}{Ŗ^{n}f(ʐ)} \prec \boldsymbol{q}(ʐ).$$

This complete***s*** the pr***oo***f of the***o***rem (2.6).

If $Ω̦ \neq \mathbb{C}$,is simpl***y*** connect***e***d d***o***main ,the $Ω̦ = Ṃ(\boldsymbol{U})$,for s***o***me c***o***nformal mapping $Ṃ(ʐ)$ ***of U on to*** $Ω̦$ .In this ca***s***e, the class $Ḩ_{i,2}[Ṃ(\boldsymbol{U}), \boldsymbol{q}]$ is ***w***ritten as $Ḩ_{i,2}[Ṃ, \boldsymbol{q}]$.

This lead***s*** to the follo***w***ing immediat***e*** consequenc***e*** of theor***e***m (2.6) ,is stated belo***w*** .

Theorem(2.7): Let $\acute{\emptyset} \in \mathrm{H}_{i,1}[\Omega, \boldsymbol{q}]$.If the functions $\boldsymbol{f} \in \boldsymbol{A}$,*and* $\boldsymbol{q} \in \boldsymbol{Q_1}$,satisfy the following conditions (2.29) ,and

$$\acute{\emptyset}\left(\frac{R^{n+1}f(z)}{R^{n}f(z)}, \frac{R^{n+2}f(z)}{R^{n+1}f(z)}, \frac{R^{n+3}f(z)}{R^{n+2}f(z)}, \frac{R^{n+4}f(z)}{R^{n+3}f(z)}; z\right) \prec M(z),$$

then

$$\frac{R^{n+1}f(z)}{R^{n}f(z)} \prec \boldsymbol{q}(z), \quad (z \in \boldsymbol{U}).$$

Definition(2.6): Let Ω be a set in $\mathbb{C}$, $\boldsymbol{q} \in \boldsymbol{Q_0} \cap \boldsymbol{A_0}$ *with* $\boldsymbol{q}'(z) \neq \boldsymbol{0}$ .The class $\mathrm{H}'_i[\Omega, \boldsymbol{q}]$ of admissible functions consists of those functions

$$\acute{\emptyset} : \mathbb{C}^4 \times \underline{\boldsymbol{U}} \to \mathbb{C}$$

That satisfy the following admissibility conditions:

$$\emptyset(a, b, \varsigma, d) \in \Omega,$$

whenever $a = \boldsymbol{q}(z)$, $b = \boldsymbol{\delta} z \boldsymbol{q}'(z)$

$$\boldsymbol{Re}\left(\frac{\varsigma - (\boldsymbol{1}-\boldsymbol{\delta})[\boldsymbol{2}b - (\boldsymbol{1}-\boldsymbol{\delta})a]}{\boldsymbol{\delta}^2 b - \boldsymbol{\delta}^2(\boldsymbol{1}-\boldsymbol{\delta})a}\right) \geq \frac{\boldsymbol{1}}{h}\boldsymbol{Re}\left(\frac{\varepsilon \boldsymbol{q}''(\varepsilon)}{\boldsymbol{q}'(\varepsilon)} + \boldsymbol{1}\right)$$

and

$$\boldsymbol{Re}\left(\frac{d - \boldsymbol{3}\varsigma - (\boldsymbol{1}-\boldsymbol{\delta})[(\boldsymbol{1}-\boldsymbol{\delta})b + (\boldsymbol{1}+\boldsymbol{\delta})a] + \boldsymbol{2}(\boldsymbol{2}-\boldsymbol{\delta})b}{\boldsymbol{\delta}^2 b - \boldsymbol{\delta}^2(\boldsymbol{1}-\boldsymbol{\delta})a}\right) \geq \frac{\boldsymbol{1}}{h^2}\boldsymbol{Re}\left(\frac{\varepsilon^2 \boldsymbol{q}'''(\varepsilon)}{\boldsymbol{q}'(\varepsilon)}\right),$$

($\boldsymbol{where}$ $z \in \boldsymbol{U}, \varepsilon \in \boldsymbol{\partial U}|\boldsymbol{E}(\boldsymbol{q})$,$h \geq \boldsymbol{2}$).

Theorem(2.8): Let $\acute{\emptyset} \in \mathrm{H}'_i[\Omega, \boldsymbol{q}]$.If the function $\boldsymbol{f} \in \boldsymbol{A}$, and $R^{n}\boldsymbol{f}(z) \in \boldsymbol{Q_0}$, ***and*** $\boldsymbol{q} \in \hat{W}_{\boldsymbol{0}}$, ***with*** $\boldsymbol{q}'(z) \neq \boldsymbol{0}$, ,satisfy the following conditions :

$$Re\left(\frac{ȥ\boldsymbol{q}''(ȥ)}{\boldsymbol{q}'(ȥ)}\right) \geq \mathbf{0}, \qquad \left| \frac{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ȥ)}{\boldsymbol{q}'(ȥ)} \right| \leq ḩ, \qquad (\mathbf{2.38})$$

and

$$Ǿ(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ȥ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ȥ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ȥ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ȥ); ȥ),$$

is univalent in U, then

$$Ɽ \subset \{Ǿ(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ȥ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ȥ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ȥ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ȥ); ȥ): ȥ \in \boldsymbol{U}\}, (\mathbf{2.39})$$

implies that $\boldsymbol{q}(ȥ) \prec Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ȥ), \quad (ȥ \in \boldsymbol{U}).$

<u>Proof</u>: L*e*t the functi*o*n ᵽ(ȥ) be defin*e*d by (2.3) and Ǿ by (2.9) .Since $Ḫ'_{\boldsymbol{i}}[Ɽ, \boldsymbol{q}]$ .Fr*o*m (2.10) and (2.39) ,we have

$$Ɽ \subset \{Ķ(ᵽ(ȥ), ȥ\, ᵽ'(ȥ), ȥ^{\mathbf{2}}ᵽ''(ȥ), ȥ^{\mathbf{3}}ᵽ'''(ȥ); ȥ): ȥ \in \boldsymbol{U}\}.$$

From (2.9) ,we see that th*e* admissibilit*y* c*o*ndition for $Ķ \in Ḫ'_{\boldsymbol{i}}[Ɽ, \boldsymbol{q}]$ i*n* definiti*o*n (2.6) is equival*e*nt to th*e* admissibilit*y* for $Ǿ \in Ḫ'_{\boldsymbol{n}}[Ɽ, \boldsymbol{q}]$ a*s* giv*e*n in definiti*o*n (1.4) *w*ith $\boldsymbol{n} = \mathbf{2}$.

Hence $Ǿ \in Ḫ'_{\mathbf{2}}[Ɽ, \boldsymbol{q}]$ and b*y* using (2.39) and lemma (1.5), *w*e find that

$$\boldsymbol{q}(ȥ) \prec Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ȥ) \quad (ȥ \in \boldsymbol{U}).$$

This c*o*mpletes the pr*oo*f of the the*o*rem (2.8) .

If $Ɽ \neq \mathbb{C}$,is simpl*y* –connect*e*d d*o*main ,th*e*n $Ɽ = Ṃ(\boldsymbol{U})$,for s*o*me conf*o*rmal mapping $Ṃ(ȥ)$ ***of U on to*** Ɽ .In this cas*e* th*e* class $Ḫ'_{\boldsymbol{i}}[Ṃ(\boldsymbol{U}), \boldsymbol{q}]$ is *w*ritten as $Ḫ'_{\boldsymbol{i}}[Ṃ, \boldsymbol{q}]$.

This lead*s* to the follo*w*ing immediat*e* consequ*e*nce of the*o*rem (2.8) is stat*e*d belo*w*.

Theorem(2.9): L*e*t $\acute{\emptyset} \in Ḫ'_i[Ṃ, \boldsymbol{q}]$ $\boldsymbol{and}$ .If the functi*o*n $\boldsymbol{f} \in \boldsymbol{A}$, and $Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ) \in \boldsymbol{Q_0}$, $\boldsymbol{and\ q} \in \hat{W}_{\boldsymbol{0}}$, $\boldsymbol{with\ q'}(ʓ) \neq \boldsymbol{0}$, ,satisf***y*** the follo***w***ing c***o***ndition***s*** (2.38) and the functi***o***n

$$\acute{\emptyset}(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ); ʓ)$$

is univalent in *U*, then

$$Ṃ(ʓ) \prec \acute{\emptyset}(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ); ʓ), \qquad (\boldsymbol{2.40})$$

implies that $\boldsymbol{q}(ʓ) \prec Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ) \quad (ʓ \in \boldsymbol{U})$.

Theorem (2.8) and (2.9) ,can onl***y*** be used to ***o***btain sub***o***rdination f***o***r the third-***o***rder differ***e***ntial super***o***rdination of th***e*** f***o***rm (2.39) or (2.40). The follo***w***ing the***o***rem giv***e***n the ***e***xistence of the b***e***st sub***o***rdinant of (2.40) for ***s***uitabl***e*** $\acute{\emptyset}$.

Theorem(2.10): L***e***t the functi***o***n Ṃ be unival***e***nt in U .Also $Ķ: \mathbb{C}^{\boldsymbol{4}} \times \boldsymbol{U} \to \mathbb{C}$,and $\emptyset$ giv***e***n b***y*** (2.9) .Suppose that follo***w***ing differential ***e***quation:

$$Ķ(ᵽ(ʓ), ʓ\,ᵽ'(ʓ), ʓ^{\boldsymbol{2}}ᵽ''(ʓ), ʓ^{\boldsymbol{3}}ᵽ'''(ʓ); ʓ) = Ṃ(ʓ), \quad (\boldsymbol{2.41})$$

has a solution $\boldsymbol{q}(ʓ) \in \acute{\emptyset}_{\boldsymbol{0}}$ *.If* $\boldsymbol{f} \in \boldsymbol{A}$ *,and* $Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ) \in \boldsymbol{Q_0}$, $\boldsymbol{and\ if\ q} \in \hat{W}_{\boldsymbol{0}}$, $\boldsymbol{with\ q'}(ʓ) \neq \boldsymbol{0}$, *satisfying the condition (2.38) ,and if the function*

$$\acute{\emptyset}(\,Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ); ʓ) \boldsymbol{is\ analytic\ in\ U},$$

then

$$Ṃ(ʓ) \prec \acute{\emptyset}(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ); ʓ),$$

implies that $\boldsymbol{q}(ʓ) \prec Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ), \quad (ʓ \in \boldsymbol{U})$

and q(ʓ) is the b***e***st sub***o***rdinant.

<u>Proof</u> : B***y*** appl***y***ing theorem (2.8) and (2.9) , ***w***e deduc***e*** that q is a sub***o***rdinant of (2.40) .Since q ***s***atisfies (2.41) ,it is als***o*** a ***s***olution of (2.40) and ther***e***fore, q will be sub***o***rdinant by all ***s***ubordinants .Hence q is th***e*** be***s***t ***s***ubordinant .This c***o***mpletes the pr***oo***f of the the***o***rem (2.10) .

<u>Definition(2.7):</u> Let Ω be a ***s***et in ℂ, $q \in \hat{W}_1$ $\boldsymbol{with}\ q'(z) \neq 0$ .The class $H'_{i,1}[\Omega, q]$ of admissibl***e*** function c***o***nsists of those functi***o***ns $\acute{\phi} : \mathbb{C}^4 \times \underline{U} \to \mathbb{C}$,ww***w***hich satisf***y*** the follo***w***ing admissibilit***y*** conditions:

$$\phi(a, b, \varsigma, d; z) \in \Omega,$$

whenever $a = q(z)$, $b = \delta z q'(z) + (1-\delta) q(z)$,

$$Rr\left(\frac{\varsigma - 2b - (1-2\delta)a}{\delta(b-a)}\right) \geq \frac{1}{h} Re\left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)} + 1\right),$$

and

$$Re\left(\frac{d - (3+3\delta)\varsigma - \big([\delta(3+\delta)+3] + (3+3\delta)(2-\delta)\big)b}{\delta^2(b-a)} - \frac{[(3+3\delta)(1-\delta) - \delta(3+\delta) - 2]a}{\delta^2(b-a)}\right) \geq \frac{1}{h^2} Re\left(\frac{\varepsilon^2 q'''(\varepsilon)}{q'(\varepsilon)}\right),$$

$(\boldsymbol{where}\ \ z \in U, \varepsilon \in \partial U \backslash E(q)$,$h \geq 2$).

<u>Theorem(2.11):</u> L***e***t $\acute{\phi} \in H_{i,1}[\Omega, q]$.If the functi***o***n $f \in A$, and $\frac{R^{n+1} f(z)}{z} \in Q_1$, $\boldsymbol{and}\ q \in \hat{W}_1$, $\boldsymbol{with}\ q'(z) \neq 0$, ,satisf***y*** the follo***w***ing c***o***nditions :

$$Re\left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)}\right) \geq 0, \quad \left| \frac{R^{n+1} f(z)}{z q'(z)} \right| \leq h, \qquad (2.42)$$

and

the function $\acute{\phi}\left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z\right)$

is univalent in $\boldsymbol{U}$, then

$$Ọ \subset \left\{ \acute{\emptyset}\left(\frac{Ṛ^{\boldsymbol{n}}\boldsymbol{f}(ʒ)}{ʒ}, \frac{Ṛ^{\boldsymbol{n+1}}\boldsymbol{f}(ʒ)}{ʒ}, \frac{Ṛ^{\boldsymbol{n+2}}\boldsymbol{f}(ʒ)}{ʒ}, \frac{Ṛ^{\boldsymbol{n+3}}\boldsymbol{f}(ʒ)}{ʒ}; ʒ\right) : ʒ \in \boldsymbol{U} \right\}, \quad (\mathbf{2.43})$$

implies that $\frac{Ṛ^{\boldsymbol{n}}\boldsymbol{f}(ʒ)}{ʒ} \prec \boldsymbol{q}(ʒ), \ (ʒ \in \boldsymbol{U}).$

<u>Proof</u>: L***e***t the functi***o***n $ᵽ(ʒ)$ be defin***e***d b***y*** (2.17) and Ķ by (2.23) .Sinc***e*** $\acute{\emptyset} \in Ḥ'_{\boldsymbol{i,1}}[Ọ, \boldsymbol{q}]$ .We find from (2.24) and (2.43) ,we have

$$Ọ \subset \{Ķ(ᵽ(ʒ), ʒ\, ᵽ'(ʒ), ʒ^{\mathbf{2}} ᵽ''(ʒ), ʒ^{\mathbf{3}} ᵽ'''(ʒ); ʒ) : ʒ \in \boldsymbol{U}\}.$$

Fr***o***m the equati***o***ns (2.21) and (2.22) , ***w***e see that the admissibilit***y*** c***o***ndition for $\acute{\emptyset} \in Ḥ'_{\boldsymbol{i,1}}[Ọ, \boldsymbol{q}]$ in d***e***finition (2.7) is equivalen***t*** t***o*** the admissibilit***y*** for $\acute{\emptyset} \in Ḥ'_{\boldsymbol{n}}[Ọ, \boldsymbol{q}]$ as giv***e***n in definiti***o***n (1.4) ***w***ith n=2.

Hence $\acute{\emptyset} \in Ḥ'_{\mathbf{2}}[Ọ, \boldsymbol{q}]$ and b***y*** u***s***ing (2.43) and lemma (1.5) ,we find that

$$\boldsymbol{q}(ʒ) \prec \frac{Ṛ^{\boldsymbol{n+1}}\boldsymbol{f}(ʒ)}{ʒ} \quad (ʒ \in \boldsymbol{U}).$$

This c***o***mpletes the pr***oo***f of the the***o***rem (2.11) .

If $Ọ \neq \mathbb{C}$,i***s*** simpl***y*** –connect***e***d d***o***main ,th***e***n $Ọ = Ṃ(\boldsymbol{U})$,f***o***r som***e*** conf***o***rmal mapping $Ṃ(ʒ)$ ***of U on to*** $Ọ$ .In thi***s*** ca***s***e the class $Ḥ'_{\boldsymbol{i,1}}[Ṃ(\boldsymbol{U}), \boldsymbol{q}]$ is ***w***ritten as $Ḥ'_{\boldsymbol{i,1}}[Ṃ, \boldsymbol{q}]$.

This lead***s*** to the follo***w***ing immediat***e*** consequenc***e*** of the***o***rem (2.11) is:

<u>Theorem(2.12):</u> *Let* $\acute{\emptyset} \in Ḥ'_{\boldsymbol{i,1}}[Ṃ, \boldsymbol{q}]$ ***and*** .If the function $\boldsymbol{f} \in \boldsymbol{A}$, *and* $\frac{Ṛ^{\boldsymbol{n}}\boldsymbol{f}(ʒ)}{ʒ} \in \boldsymbol{Q_1}$, ***and*** $\boldsymbol{q} \in \hat{W}_{\mathbf{1}}$, ***with*** $\boldsymbol{q}'(ʒ) \neq \mathbf{0}$, ,satisf**y**ing the following conditions (2.42) and the function

$$\acute{\emptyset}\left(\frac{Ṛ^{\boldsymbol{n}}\boldsymbol{f}(ʒ)}{ʒ}, \frac{Ṛ^{\boldsymbol{n+1}}\boldsymbol{f}(ʒ)}{ʒ}, \frac{Ṛ^{\boldsymbol{n+2}}\boldsymbol{f}(ʒ)}{ʒ}, \frac{Ṛ^{\boldsymbol{n+3}}\boldsymbol{f}(ʒ)}{ʒ}; ʒ\right)$$

is univalent in $\boldsymbol{U}$, then

$$Ṃ(ẓ) \prec \acute{\emptyset}\left(\frac{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ẓ)}{ẓ}, \frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ẓ)}{ẓ}, \frac{Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ẓ)}{ẓ}, \frac{Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ẓ)}{ẓ}; ẓ\right),$$

implies that $\boldsymbol{q}(ẓ) \prec \frac{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ẓ)}{ẓ}$, $(ẓ \in \boldsymbol{U})$.

Definition(2.8): Let $Ọ$ be a set in $\mathbb{C}$ and $\boldsymbol{q} \in \hat{W}_{\boldsymbol{1}}$ $\boldsymbol{with}\ \boldsymbol{q}'(ẓ) \neq \boldsymbol{0}$ .The class $Ḥ'_{\boldsymbol{i},\boldsymbol{1}}[Ọ, \boldsymbol{q}]$ of admissible functions consists of those functions $\acute{\emptyset} : \mathbb{C}^{\boldsymbol{4}} \times \underline{\boldsymbol{U}} \to \mathbb{C}$, which satisfy the following admissibility conditions:

$$\emptyset(ą, ɓ, \boldsymbol{\varsigma}, ȡ; ẓ) \in Ọ,$$

whenever $ą = \boldsymbol{q}(ẓ)$, $ɓ = \frac{\boldsymbol{\delta}ẓ\boldsymbol{q}'(ẓ)+(\boldsymbol{q}(ẓ))^{\boldsymbol{2}}}{\boldsymbol{q}(ẓ)}$,

$$\boldsymbol{R}\left(\frac{\boldsymbol{\varsigma}ɓ + \boldsymbol{2}ą^{\boldsymbol{2}} - \boldsymbol{3}ąɓ}{\boldsymbol{\delta}(ɓ - ą)}\right) \geq \frac{\boldsymbol{1}}{ḫ}\boldsymbol{R}\left(\frac{ɛ\boldsymbol{q}''(ɛ)}{\boldsymbol{q}'(ɛ)} + \boldsymbol{1}\right),$$

and

$$\begin{aligned}\boldsymbol{Re}([(ɓ\boldsymbol{\varsigma} - ąɓ + ą^{\boldsymbol{2}} + \boldsymbol{\delta}ą][ɓȡ - ɓ\boldsymbol{\varsigma} + ąɓ - ą^{\boldsymbol{2}} - \boldsymbol{\delta}ą] - ɓ^{\boldsymbol{3}}(\boldsymbol{\varsigma} - ɓ)\\ - [ɓ(\boldsymbol{\varsigma} - \boldsymbol{\delta} - \boldsymbol{3}ą) + ą(\boldsymbol{\delta} + \boldsymbol{2}ą)]\Bigg[ɓ(\boldsymbol{5}ɓ - \boldsymbol{5}ą - \boldsymbol{\varsigma} + \boldsymbol{2\delta})\\ + ą(\boldsymbol{\delta} + \boldsymbol{2}ą) + \frac{ą^{\boldsymbol{2}}}{\boldsymbol{\delta}^{\boldsymbol{3}}}[(ɓ(\boldsymbol{\varsigma} - \boldsymbol{3}ą - \boldsymbol{\delta}) + ą(\boldsymbol{\delta} + \boldsymbol{2}ą)]^{\boldsymbol{2}}\\ - (ɓ - ą)[ąɓ(\boldsymbol{\delta}^{\boldsymbol{2}} - ą + \boldsymbol{4}ɓ) + ɓ(\boldsymbol{\delta}^{\boldsymbol{2}} + \boldsymbol{\delta}ɓ - \boldsymbol{2}ɓ^{\boldsymbol{2}})\\ - ą(\boldsymbol{\delta} + \boldsymbol{\delta}ą + \boldsymbol{\delta}^{\boldsymbol{2}}ą - ą^{\boldsymbol{2}})]\Bigg] * [\boldsymbol{\delta}^{\boldsymbol{2}}(ɓ - ą)]^{-\boldsymbol{1}}) \geq \frac{\boldsymbol{1}}{ḫ^{\boldsymbol{2}}}\boldsymbol{Re}\left(\frac{ɛ^{\boldsymbol{2}}\boldsymbol{q}'''(ɛ)}{\boldsymbol{q}'(ɛ)}\right),\end{aligned}$$

$(\boldsymbol{where}\ \ ẓ \in \boldsymbol{U}, ɛ \in \boldsymbol{\partial U}|\boldsymbol{E}(\boldsymbol{q})\ ,ḫ \geq \boldsymbol{2}\ )$.

Theorem(2.13): Let $\acute{\emptyset} \in Ḥ_{\boldsymbol{i},\boldsymbol{2}}[Ọ, \boldsymbol{q}]$.If the function $\boldsymbol{f} \in \boldsymbol{A}$, and $\frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ẓ)}{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ẓ)} \in$

$\boldsymbol{Q_1}$, $\boldsymbol{and}\ \boldsymbol{q} \in \hat{W}_{\boldsymbol{1}}$, $\boldsymbol{with}\ \boldsymbol{q}'(ẓ) \neq \boldsymbol{0}$, ,satisfy the following conditions :

$$\boldsymbol{Re}\left(\frac{\varepsilon \boldsymbol{q}''(\varepsilon)}{\boldsymbol{q}'(\varepsilon)}\right) \geq \mathbf{0}, \qquad \left|\ \frac{R^{n+1}\boldsymbol{f}(z)}{R^{n}\boldsymbol{f}(z)}\ \right| \leq h, \qquad (\mathbf{2.44})$$

and

$$\phi\left(\frac{R^{n+1}\boldsymbol{f}(z)}{R^{n}\boldsymbol{f}(z)},\frac{R^{n+2}\boldsymbol{f}(z)}{R^{n+1}\boldsymbol{f}(z)},\frac{R^{n+3}\boldsymbol{f}(z)}{R^{n+2}\boldsymbol{f}(z)},\frac{R^{n+4}\boldsymbol{f}(z)}{R^{n+3}\boldsymbol{f}(z)};z\right)$$

is univalent in $\boldsymbol{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{R^{n+1}\boldsymbol{f}(z)}{R^{n}\boldsymbol{f}(z)},\frac{R^{n+2}\boldsymbol{f}(z)}{R^{n+1}\boldsymbol{f}(z)},\frac{R^{n+3}\boldsymbol{f}(z)}{R^{n+2}\boldsymbol{f}(z)},\frac{R^{n+4}\boldsymbol{f}(z)}{R^{n+3}\boldsymbol{f}(z)};z\right): z \in \boldsymbol{U}\right\}, (\mathbf{2.45})$$

implies that $\boldsymbol{q}(z) \prec \frac{R^{n+1}\boldsymbol{f}(z)}{R^{n}\boldsymbol{f}(z)}, \quad (z \in \boldsymbol{U}).$

<u>Proof</u>: Let the functi***o***n $p(z)$ be defin***e***d b***y*** (2.30) and K b***y*** (2.36) .Since $\phi \in H'_{i,2}[\Omega, \boldsymbol{q}]$ .We find from (2.37) and (2.45) ,we have

$$\Omega \subset \{K(p(z), z\, p'(z), z^2 p''(z), z^3 p'''(z); z): z \in \boldsymbol{U}\}.$$

Fr***o***m the equations (2.34) and (2.35) , ***w***e see tha***t*** the admissibilit***y*** conditi***o***n for $\phi \in H'_{i,2}[\Omega, \boldsymbol{q}]$ in definiti***o***n (2.8) i***s*** equivalen***t*** to th***e*** admissibilit***y*** f***o***r $K \in H'_{n}[\Omega, \boldsymbol{q}]$ as giv***e***n in definiti***o***n (1.4) ***w***ith $\boldsymbol{n} = \mathbf{2}.$

Henc***e*** $K \in H'_{2}[\Omega, \boldsymbol{q}]$ and b***y*** using (2.44) and lemma (1.5) , ***w***e find that

$$\boldsymbol{q}(z) \prec \frac{R^{n+1}\boldsymbol{f}(z)}{R^{n}\boldsymbol{f}(z)} \quad (z \in \boldsymbol{U}).$$

This complete***s*** the pr***oo***f ***o***f the theor***e***m (2.13) .

If $\Omega \neq \mathbb{C}$, is simpl***y*** –connect***e***d d***o***main ,then $\Omega = M(\boldsymbol{U})$,for s***o***me c***o***nformal mapping $M(z)$ ***of U on to*** Ω .In thi***s*** cas***e*** the class $H'_{i,2}[M(\boldsymbol{U}), \boldsymbol{q}]$ is ***w***ritten as $H'_{i,2}[M, \boldsymbol{q}]$.

This lead $\boldsymbol{s}$ to the follo$\boldsymbol{w}$ing immediat$\boldsymbol{e}$consequenc$\boldsymbol{e}$ $\boldsymbol{o}$f theor$\boldsymbol{e}$m (2.13) is:

Theorem(2.14): Let $\acute{\emptyset} \in Ḥ'_{i,2}[Ṃ, \boldsymbol{q}]$ $\boldsymbol{and}$ .If the functi$\boldsymbol{o}$n $\boldsymbol{f} \in \boldsymbol{A}$, and $\frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ)} \in \boldsymbol{Q_1}$, $\boldsymbol{and}\ \boldsymbol{q} \in \hat{W}_{\boldsymbol{1}}$, $\boldsymbol{with}\ \boldsymbol{q}'(ʓ) \neq \boldsymbol{0}$, , $\boldsymbol{s}$atisfying the follo$\boldsymbol{w}$ing condition$\boldsymbol{s}$ (2.44) and the f$\boldsymbol{un}$ction

$$\acute{\emptyset}\left(\frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ)}, \frac{Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ)}, \frac{Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ)}, \frac{Ŗ^{\boldsymbol{n+4}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ)}; ʓ\right)$$

is univalent in $\boldsymbol{U}$, then

$$Ṃ(ʓ) \prec \acute{\emptyset}\left(\frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ)}, \frac{Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ)}, \frac{Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ)}, \frac{Ŗ^{\boldsymbol{n+4}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ)}; ʓ\right),$$

implies that $\boldsymbol{q}(ʓ) \prec \frac{Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ)}{Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ)}, \quad (ʓ \in \boldsymbol{U}).$

By combining theorem (2.2) and (2.9) , $\boldsymbol{w}$e obtain follo$\boldsymbol{w}$ing sand$\boldsymbol{w}$ich-typ$\boldsymbol{e}$ theorem .

Theorem (2.15): Let $Ṃ_{\boldsymbol{1}}$ $\boldsymbol{and}\ \boldsymbol{q_1}$ be anal$\boldsymbol{y}$tic func$\boldsymbol{t}$ions in U .Als $\boldsymbol{o}$ let $Ṃ_{\boldsymbol{2}}$ b $\boldsymbol{e}$ univalen$\boldsymbol{t}$ functi$\boldsymbol{o}$n in *U* ,and $\boldsymbol{q_2} \in \boldsymbol{Q_1}\ \boldsymbol{with}\ \boldsymbol{q_1}(\boldsymbol{0}) = \boldsymbol{q_2}(\boldsymbol{0}) = \boldsymbol{1}$ and $\acute{\emptyset} \in Ḥ'_{\boldsymbol{i}}[Ṃ_{\boldsymbol{1}}, \boldsymbol{q_1}] \cap Ḥ'_{\boldsymbol{i},}[Ṃ_{\boldsymbol{2}}, \boldsymbol{q_2}]$.If the function $\boldsymbol{f} \in \boldsymbol{A}$, *with* $Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ) \in \boldsymbol{Q_0} \cap Ạ_{\boldsymbol{0}}$, *and the function*

$$\acute{\emptyset}(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ); ʓ),$$

is univalent in $\boldsymbol{U}$,and if the c$\boldsymbol{o}$ndition (2.1) and (2.38) are $\boldsymbol{s}$atisfied,

then

$$Ṃ_{\boldsymbol{1}}(ʓ) \prec \acute{\emptyset}(Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+1}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+2}}\boldsymbol{f}(ʓ), Ŗ^{\boldsymbol{n+3}}\boldsymbol{f}(ʓ); ʓ) \prec Ṃ_{\boldsymbol{2}}(ʓ),$$

*i*mplies that $\boldsymbol{q_1}(ʓ) \prec Ŗ^{\boldsymbol{n}}\boldsymbol{f}(ʓ) \prec \boldsymbol{q_2}(ʓ), \quad (ʓ \in \boldsymbol{U}). \qquad (\boldsymbol{2.46})$

On th$\boldsymbol{e}$ other hand ,we combin$\boldsymbol{e}$ theorem (2.5) and (2.12) ,we ob$\boldsymbol{t}$ain the follo$\boldsymbol{w}$ing Sand$\boldsymbol{w}$ich-typ$\boldsymbol{e}$ theorem

<u>Theorem (2.16):</u> Let M_1 ***and*** q_1 be analytic functions in U .Also let M_2 be univalent functi o n in U ,and $q_2 \in Q_1$ ***with*** $q_1(0) = q_2(0) = 1$ and $\phi \in H'_{i,1}[M_1, q_1] \cap H'_{i,1}[M_2, q_2]$.If the function $f \in A$, *with* $\frac{R^n f(z)}{z} \in Q_1 \cap A_1$, *and the function*

$$\phi\left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z\right),$$

is univalent in U,and if the conditions (2.15) and (2.42) are satisfied,

then

$$M_1(z) \prec \phi\left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z\right) \prec M_2(z),$$

implies that $$q_1(z) \prec \frac{R^n f(z)}{z} \prec q_2(z), \quad (z \in U). \qquad (2.47)$$

<u>Theorem (2.17):</u> Let M_1 ***and*** q_1 be analytic functions in U .Also let M_2 b e univalent functi o n in U ,and $q_2 \in Q_1$ ***with*** $q_1(0) = q_2(0) = 1$ and $\phi \in H'_{i,2}[M_1, q_1] \cap H'_{i,2}[M_2, q_2]$. If the function $f \in A$, *with* $\frac{R^{n+1} f(z)}{R^n f(z)} \in Q_1 \cap A_{p1}$, and the function

$$\phi\left(\frac{R^{n+1} f(z)}{R^n f(z)}, \frac{R^{n+2} f(z)}{R^{n+1} f(z)}, \frac{R^{n+3} f(z)}{R^{n+2} f(z)}, \frac{R^{n+4} f(z)}{R^{n+3} f(z)}; z\right)$$

is univalent in U, and if the condition (2.28) and (2.44) are satisfied,

then

$$M_1(z) \prec \phi\left(\frac{R^{n+1} f(z)}{R^n f(z)}, \frac{R^{n+2} f(z)}{R^{n+1} f(z)}, \frac{R^{n+3} f(z)}{R^{n+2} f(z)}, \frac{R^{n+4} f(z)}{R^{n+3} f(z)}; z\right) \prec M_2(z),$$

implies that $$q_1(z) \prec \frac{R^{n+1} f(z)}{R^n f(z)} \prec q_2(z), \quad (z \in U). \qquad (2.48)$$

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